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RELATIVE FLATNESS AND ASYMPTOTIC BEHAVIOUR

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Abstract. Let R be a commutative ring and let I be an ideal of R. Let M be a Noetherian R-modules. Let F be an R-module which is flat relative to M. Suppose M' is a submodule of M and let $T = F \otimes_R -$. Then it is shown that the sequences of sets

$$Ass_R(T(M)/I^n(T(M'))) \text{ and } Ass_R(I^n(T(M))/I^n(T(M'))), n \in \mathbf{N}$$

are ultimately constant.

1. Introduction

Throughout of this paper R will denote a commutative ring (with a nonzero identity), and $\mathbf{N}$ is the set of all positive integers.

Let I be an ideal of R and M be a Notherian R-module. If M' is a submodule of M it follows from [3]that both sequences of sets $Ass_R(M/I^nM')$ and $Ass_R(I^nM/I^nM')$, $n \in \mathbf{N}$ are ultimately constant.

Let R be a commutative Noetherian ring and let I be an ideal of R. Further assume that F is a flat R-module. Set $T = F \otimes_R -$. In [4], S. Yassemi showed that the $Ass_R(T(M))$ can be specified in terms of $Ass_R(M)$ and $Coass_R(F)$. Also it is shown that the sequences of sets

$$Ass_R(T(M/I^nM)) \text{ and } Ass_R(T(I^nM/I^{n+1}M)), n \in \mathbf{N}$$

are ultimately constant.

Now let M be an R-module and let the zero submodule of M have a primary decomposition. Let F be an R-module which is flat relative to M. Set $E = E(\bigoplus_{P \in Max(R)} R/P)$ and $F^\vee = Hom_R(F, E)$. In this paper we will generalize the results mentioned in last paragraph by showing that $M \otimes_R F$ has a primary decomposition and its weakly associated primes can be specified (see 2.1) in terms of $W.Ass_R(M)$ and $Ass_R(F^\vee)$ under an additional property. When R is a quasi-semi local ring, $W.Ass_R(M \otimes_R F)$ is specified (see 2.4) in

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terms of $W.Ass_R(M)$ and $Coass_R(F)$. Also when R is a commutative Noetherian ring, $Ass_R(M \otimes_R F)$ can be specified (see 2.3, 2.6) in terms of $Ass_R(M)$ and $Coass_R(F)$ without any restriction. Finally it is shown that if M is a Noetherian R-module then for every submodule M' of M, the sequences of sets

$$Ass_R((M \otimes_R F)/I^n(M' \otimes_R F)), n \in \mathbf{N}$$

and

$$Ass_R(I^n(M \otimes_R F)/I^n(M' \otimes_R F)), n \in \mathbf{N}$$

are ultimately constant (see 2.8). We recall that F is flat relative to M (or F is M-flat) if and only if for any submodule N of M, the homomorphism $F \otimes_R N \to F \otimes_R M$ is monic (see [1]).

2. Main results

Theorem 2.1. *Let M be an R-module and let F be an R-module which is flat relative to M. Further assume that the zero submodule of M has a primary decomposition and $Ass_R(F^\vee) = W.Ass_R(F^\vee)$, where $F^\vee = Hom_R(F, E)$ and $E = E(\bigoplus_{P \in Max(R)} R/P)$. Then $M \otimes_R F$ has a primary decomposition and we have*

$$W.Ass_R(M \otimes_R F) = \{P \in W.Ass_R(M) : P \subseteq Q \ for \ some \ Q \in W.Ass_R(F^\vee)\}.$$

Corollary 2.2. *Let the situation be as in 2.1 and let F be flat R-module. Then we have*

$$W.Ass_R(M \otimes_R F) = \{P \in W.Ass_R(M) : P \subseteq Q \ for \ some \ Q \in W.Ass_R(F^\vee)\}.$$

Corollary 2.3. *Let R be a commutative Noetherian ring and let M be an R-module. Let F be an R-module which is flat relative to M. Further assume that the zero submodule of M has a primary decomposition. Then $M \otimes_R F$ has a primary decomposition and we have*

$$Ass_R(M \otimes_R F) = \{P \in Ass_R(M) : P \subseteq Q \ for \ some \ Q \in Coass_R(F)\}.$$

Corollary 2.4. *Let R be a semi- quasi local ring and let M be an R-module with the property that its zero submodule has a primary decomposition. Further assume that F is an R-module which is flat relative to M and that $W.Coass_R(F) = Coass_R(F)$ (This is true, for example, when each element of $W.cass_R(M)$ is finitely generated by [[6] , (2.4)]). Then we have*

$$W.Ass_R(M \otimes_R F) = \{P \in W.Ass_R(M) : P \subseteq Q \ for \ some \ Q \in Coass_R(F)\}.$$

Corollary 2.5. *Let M be an R-module and let F be an R-module which is flat relative to M. Further assume that the zero submodule of M has a primary decomposition and $Ass_R(F^\vee) = W.Ass_R(F^\vee)$, where $E = E(\bigoplus_{P \in Max(R)} R/P)$ and $F^\vee = Hom_R(F, E)$. Then $(M \otimes_R F) \neq 0$ if and only if there exits $P \in W.Ass_R(M)$ such that $P \subseteq Q$ for some $Q \in Ass_R(F^\vee)$. Further,*

$$W.Ass_R(M \otimes_R F) = \bigcup_{P \in W.Ass_R(M)} W.Ass_R(F/PF).$$

Theorem 2.6. *Let R be a commutative Noetherian ring and let M be an R-module. Let F be an R-module which is flat relative to M. Then we have*

$$Ass_R(M \otimes_R F) = \{P \in Ass_R(M) : P \subseteq Q \ for\ some\ Q \in Coass_R(F)\}.$$

Corollary 2.7. *Let R be a commutative Noetherian ring let M be an R-module. let F be an R-module which is flat relative to M. Then we have*

$$Ass_R(M \otimes_R F) = \bigcup_{P \in Ass_R(M)} Ass_R(F/PF).$$

Theorem 2.8. *Let R be a commutative Noetherian ring and let M be a Noetherian R-module. Suppose F is an R-module which is flat relative to M. Let M' be a submodule of M and let $T = F \otimes_R -$. Then for an ideal I of R, the sequence of sets*

$$Ass_R(T(M)/I^n(T(M'))) \ and\ Ass_R(I^n(T(M))/I^n(T(M'))), n \in \mathbf{N}$$

are ultimately constant. If we denote the ultimate constant value of the above sequences by T_1 and T_2, then we have

$$T_1 = \{P \in As^*(I, M', M) : P \subseteq Q \ for\ some\ Q \in Coass_R(F)\}$$

and

$$T_2 = \{P \in Bs^*(I, M', M) : P \subseteq Q \ for\ some\ Q \in Coass_R(F)\}$$

Theorem 2.9. *Let M be a Noetherian R-module and suppose that F is an R-module which is flat relative to M. Let M' be a submodule of M and let $T = F \otimes_R -$. Then for an ideal I of R, the sequence of sets*

$$Ass_R(T(M)/I^n(T(M'))) \ and\ Ass_R(I^n(T(M))/I^n(T(M'))), n \in \mathbf{N}$$

are ultimately constant.

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