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THE PRIME SPECTRUM ON BL-ALGEBRAS AND MV-ALGEBRAS

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Abstract. In 1958 C. C. Chang developed an algebraic version of Łukasiewicz propositional logic and provided an algebraic proof of the completeness. The resulting algebraic system became known as an MV-algebra.
In 1998 Peter Hájek introduced the variety of BL-algebras and showed that the variety of MV-algebras actually is a subvariety of the variety of BL-algebras. In other words, any MV-algebra can be easily viewed as a special BL-algebra. Let A be an MV-algebra. We denote by $Spec(A)$ the set of prime ideals of A. $Spec(A)$ can be endowed with a spectral topology in the same manner as in the case of a commutative ring or a bounded distributive lattice. However the absence of a distributive law requiers a reformulation of the results known in ring theory. Thus it makes sense to generalize the notion of prime spectrum to BL-algebras.

1. Introduction

The theory of MV-algebras has its origin in the study of the system of infinite-valued logic originated by Lukasiewicz. The completeness of the propositional Łukasiewicz logic was first published by Rose and Rosser in 1958. An earlier proof by Wajsberg was never published. MV-algebras, therefore, stand in relation to the Łukasiewicz infinite valued logic as Boolean algebras stand in relation to classical 2-valued logic. Boolean algebras, of course, have not stayed glued to their origin in logic, their uses showing up in other areas of mathematics. Moreover there have been extensive investigations concerning their structure. The same can be said about MV-algebras, that is their connecting to other areas of mathematics and investigations of their intrinsic structure.
BL-algebras have been invented recently by Peter Hájek in order to provide an algebraic proof of the completeness theorem of a class of $[0, 1]$- valued logics familiar in fuzzy logic framework and investigate many valued logic by algebraic means. In 1998, Hájek introduce a very general many valued logic, called

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Basic Logic or (BL), with the idea to formalized many valued semantic induce by the continuous t-norm on the unit real interval $[0, 1]$. This basic logic turns to be a fragment common to three important many valued logic. The Lindenbaum-tareski algebras for basic logic are called BL-algebras. Apart from their logical interest, BL-algebras have important algebraic properties and they have been intensively studied from an algebraic point of view. His motivation for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic which embodies a fragment common to some of the most important many valued logics, namely Łukasiwiewcz logic, Gödel Logic and product logic. This Basic Logic (BL for short) is proposed as the most general many-valued logic with truth values in [0,1]. The second one was to provide an algebraic means for the study of continuous t-norms on [0,1]. Let A be an MV-algebra. We denote by $Spec(A)$ the set of prime ideals of A. $Spec(A)$ can be endowed with a spectral topology in the same manner as in the case of a commutative ring or a bounded distributive lattice. Thus, if I is an ideal of A, then $U(I) = \{P \in Spec(A) | I \nsubseteq P\}$ is open in $Spec(A)$, while $V(I) = \{P \in Spec(A) | I \subseteq P\}$ is closed. Also, let $a \in A$, the open sets $U(a) = \{P \in Spec(A) | a \notin P\}$ constitute a basis for the open sets of $Spec(A)$. Topological space $Spec(A)$ is called *the prime spectrum of A*.

2. MV-algebras

Definition 2.1. An MV-algebra is a algebra $(A, \oplus, -, 0)$ with a binary operation $\oplus : A \times A \longrightarrow A$, a unary operation $- : A \longrightarrow A$ and a constant 0 satisfying the following equations for each $x, y, z \in A$:
(1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(2) $x \oplus y = y \oplus x$;
(3) $x \oplus 0 = x$;
(4) $\overline{\overline{x}} = x$;
(5) $x \oplus \overline{0} = \overline{0}$;
(6) $\overline{(\overline{x} \oplus y)} \oplus y = \overline{(\overline{y} \oplus x)} \oplus x$.

Definition 2.2. An ideal of an MV-algebra A is a subset I of A satisfying the following conditions:
(i) $0 \in A$;
(ii) If $x \in I$, $y \in A$ and $y \leq x$ then $y \in I$;
(iii) If $x \in I$ and $y \in I$ then $x \oplus y \in I$.

Theorem 2.3. *Let J be an ideal of an MV-algebra A. Then J is a prime ideal of A iff J is a prime ideal of the underlying lattice L(A).*

Lemma 2.4. *In every MV-algebra A the natural order "$\leq$" has the following properties:*
1) $x \leq y$ *iff* $\overline{y} \leq \overline{x}$;
2) *If* $x \leq y$ *then for each* $z \in A$, $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$;
3) $x \odot y \leq z$ *iff* $x \leq \overline{y} \oplus z$.

Corollary 2.5. *The following properties hold in any MV-algebra A.*
(i) *Every proper ideal of A that contains a prime ideal is prime;*

(ii) *For each prime ideal J of A, the set* $\{I \in \daleth(A) | \ J \subseteq I\}$ *where* $\daleth(A)$ *is the set of all ideals of A, is totally ordering by inclusion.*

3. BL-algebras

Definition 3.1. [10] *A BL-algebra is an algebra* $(A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ *with four binary operations* $\wedge, \vee, \odot, \longrightarrow$ *and two constant 0,1 such that* $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice,* $(A, \odot, 1)$ *is a commutative monoid, and for all* $a, b, c \in A$*,*

$$c \leq a \longrightarrow b \ iff \ a \odot c \leq b \tag{1}$$

$$a \wedge b = a \odot (a \longrightarrow b) \tag{2}$$

$$(a \longrightarrow b) \vee (b \longrightarrow a) = 1 \tag{3}$$

Example 3.2. We introduce an example of a finite BL-algebra which is not a MV-algebra. We begin with $A = \{a, b, c, 1\}$ and define the operations as follows.

$\longrightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\vee$	0	c	a	b	1
0	0	c	a	b	1
c	c	c	a	b	1
a	a	a	a	1	1
b	b	b	1	b	1
1	1	1	1	1	1

$\wedge$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

It is easy to see that A with the above operations is a BL-algebra. We claim that for all $x \in A, \ x \neq 0, \ \overline{x} = 0$, since $\overline{0} = 1$, $\overline{1} = 1 \longrightarrow 0 = 0$, $\overline{a} = a \longrightarrow 0 = 0$, $\overline{b} = b \longrightarrow 0 = 0$, $\overline{c} = c \longrightarrow 0 = 0$. Thus $\overline{\overline{x}} = 1$ for all $x \in A, \ x \neq 0$ and $\overline{\overline{0}} = 0$. Hence A is not an MV-algebra.

Definition 3.3. A filter of a BL-algebra A is a nonempty subset $F \subseteq A$ such that for all $x, y \in A$:
(i) $x, y \in F$ implies $x \odot y \in F$;
(ii) $x \in F$ and $x \leq y$ imply $y \in F$.

4. Prime Spectrum

Let A be a nontrivial BL-algebra. For each subset X of A, we define $V(X) = \{P \in Spec(A) | X \subseteq P\}$.
The family $\{V(X)\}_{X \subseteq A}$ of subsets of $Spec(A)$ satisfies the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology* and the topological space $Spec(A)$ is called the *prime spectrum* of A.
For any $X \subseteq A$, let us denote the complement of $V(X)$ by $U(X)$. Hence, $U(X) = \{P \in Spec(A) | X \subsetneqq P\}$. It follows that the family $\{U(X)\}_{X \subseteq A}$ is the family of open sets of the zariski topology. For any $a \in A$, let us denote $V(\{a\})$ by $V(a)$ and $U(\{a\})$ by $U(a)$. Then,
$V(a) = \{P \in Spec(A) | a \in P\}$ and $U(a) = \{P \in Spec(A) | a \notin P\}$.

Proposition 4.1. *Let A be a nontrivial BL-algebra. Then*
i) the family $\{U(a)\}_{a \in A}$ is a basis for the topology of $Spec(A)$.
ii) for any $a \in A$, $U(a)$ is compact in $Spec(A)$;
iii) $Spec(A)$ is a compact T_0 topological space.
iv) $Max(A)$ is a compact Hausdorff topological space, where $Max(A)$ is the set of all maximal filters of A.

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