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FUZZY PRIMARY SUBACTS

A. A. ESTAJI

Department of Mathematics
Teacher Training University
Sabzevar, Iran. aa_estaji@yahoo.com
(Joint work with H. Zeinalian)

Abstract. In this paper the concepts of a fuzzy primary subact is given and some fundamental results are proved. Also a characterization of a fuzzy P-primary subact is given.

1. Introduction

In [3] the notion of L-fuzzy primary (P-primary) submodule of M is given in terms of fuzzy singletons. In this paper, we generalize this definition to fuzzy primary (P-primary) subact.

Throughout this paper S will denote a commutative monoid with 0. Recall that a *right S-act* is a set A together with a function $\lambda : A \times S \to A$, called the *action* of S (or the S-action) on A, such that for $a \in A$ and $s, t \in S$ (denoting $\lambda(a, s)$ by as), $a(st) = (as)t$ and $a1 = a$.

Throughout this paper all acts is *centered right S-act* (if there is a unique fixed element in A denoted θ such that $\theta s = \theta$ and $a0 = \theta$, $\forall s \in S$ and $a \in A$) and A will always denote a centered right S-act. We denote by $F(X)$ the set of all fuzzy subsets of X. For $\mu, \lambda \in F(X)$, we say $\mu \subseteq \lambda$ iff $\mu(x) \leq \lambda(x)$ for all $x \in X$. Let $\mu \in F(X)$ and $t \in [0, 1]$. Then the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called the level subset of X with respect to μ. Also we put $\mu_* = \{x \in X : \mu(x) \geq 1\}$. $x^r \in F(X)$ is called a *fuzzy point* iff $x^r(y) = 0$ for $y \neq x$, and $x^r(x) = r \neq 0$. The fuzzy point x^r is said to belong to $\mu \in F(X)$, written $x^r \in \mu$, iff $\mu(x) \geq t$. Let $\mu, \lambda \in F(X)$. Then $\mu \subseteq \lambda$ iff $x^t \in \mu$ implies $x^t \in \lambda$ for all fuzzy point $x^t \in FP(X)$. $\lambda \in F(S)$ is called a *fuzzy right (left) ideal* of S if $\lambda \circ \chi_S \subseteq \lambda$ ($\chi_S \circ \lambda \subseteq \lambda$). It is clear that μ is a fuzzy ideal of S iff $\mu(xy) \geq \mu(x) \vee \mu(y)$. We denote by $FI(S)$, the set of all fuzzy ideal of S. Throughout this paper we suppose that if λ is a fuzzy ideal of S, then $\lambda(0) = 1$.

Let I be an ideal of S. Then I is called a *prime ideal* of S if for all $a, b \in S$, $ab \subseteq I$ implies that $a \in I$ or $b \in I$. Note for ideal I of S, the notation $\sqrt{I}$ is the intersection of all prime ideal of S containing I and $\sqrt{I} = S$ if I is not contained in any prime ideal of S. In [2], it is proved that if I is an ideal I of S, then $\sqrt{I} = \{s \in S : s^n \in I \text{ for some } n \in \mathbb{N}\}$. Let I be an ideal of S. Then

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I is called a *primary ideal* of S if for all $a, b \in S$, $ab \subseteq I$ implies that $a \in I$ or $b \in \sqrt{I}$.

Let $\mu \in FI(S)$ be non-constant. Then μ is called be a *fuzzy prime ideal* of S if for any $\lambda, \gamma \in FI(S)$, $\lambda \circ \gamma \subseteq \mu$ implies that either $\lambda \subseteq \mu$ or $\gamma \subseteq \mu$.

Theorem 1.1. [4] *$\mu \in FI(S)$ is a fuzzy prime ideal of S iff for any two fuzzy points $x^r, y^s \in FP(S)$, $x^r \circ y^s \subseteq \mu$ implies that either $x^r \in \mu$ or $y^s \in \mu$.*

Let $\mu \in FI(S)$ be non-constant. Then μ is called a *fuzzy primary ideal* of S iff for any fuzzy points $x^r, y^s \in FP(S)$, $x^r \circ y^s \subseteq \mu$ implies that either $x^r \in \mu$ or $(y^s)^n \in \mu$ for some $n \in \mathbb{N}$.

We observe that any fuzzy prime ideal is a fuzzy primary ideal.

If $\mu \in FI(S)$, then we put

$$\sqrt{\mu} = \begin{cases} \bigcap_{\mu \subseteq P} P & if\ there\ is\ a\ fuzzy\ prime\ ideal\ P\ such\ that\ \mu \subseteq P \\ \chi_S & otherwise. \end{cases}$$

In [2] a proper subact B of A is said to be *primary subact* of A if for every $s \in S$ and $a \in A$, $as \in B$ implies that $a \in B$ or $s \in \sqrt{(B : A)}$. Note that if $s \in \sqrt{(B : A)}$, then there is an $n \in \mathbb{N}$ such that $s^n \in (B : A)$. It is not difficult to see that if B is a primary subact of A and $P = \sqrt{(B : A)})$, then P is a prime ideal of S. Thus if B is a primary subact of A and $P = \sqrt{(B : A)}$, we will say that B is a P-primary subact. In [2] it is proved that if I is a proper ideal of S, then $\sqrt{I} = \{t \in S : \exists n \in \mathbb{N}(t^n \in I)\}$.

2. Fuzzy subact

We now define a fuzzy S-act of A. Let A be a S-act and $\mu \in F(A)$. Then μ is called a *fuzzy right(left) S-act* of A if $\mu(as) \geq \mu(a)$ $(\mu(sa) \geq \mu(a))$ for all $a \in A$ and for all $s \in S$ and $\mu(\theta) = 1$. We denote by $FS(A)$ the set of all fuzzy right S-act of A.

Let $\mu \in F(A)$ and $\lambda \in F(S)$. Define the composition of μ and λ as follows: $(\mu \circ \lambda)(x) = \vee\{\mu(a) \wedge \lambda(s) \mid x = as$ for some $a \in A$ and $s \in S\}$ for all $x \in A$.

Proposition 2.1. *If $\mu \in FS(A)$ and $\lambda \in FI(S)$, then $\mu \circ \lambda \in FS(A)$.*

Lemma 2.2. *If $A = S$, then μ is a fuzzy S-act of A iff μ is a fuzzy ideal of S.*

Theorem 2.3. *Let $\mu \in F(A)$ and $\mu(\theta) = 1$. Then $\mu \in FS(A)$ iff μ_t is a subact of A for all $t \in Im(\mu)$.*

3. Fuzzy Primary subact

In this section we give some characterizations for fuzzy primary subacts of A. We start with some definitions of ordinary subact which will be expressed later in fuzzy context.

Let μ, $\lambda \in FS(A)$ be non-constant. Then λ is called a *fuzzy subact* of μ on A iff $\lambda \subset \mu$. If $\mu = \chi_A$ and λ be non-constant, then λ is called a fuzzy subact of A.

Let λ be a fuzzy subact of μ. Then λ is said to be a *fuzzy primary subact* of μ on A iff for any fuzzy points $s^\alpha \in FP(S)$ and $a^\beta \in FP(A)$, $a^\beta \circ s^\alpha \subseteq \lambda$ implies that $a^\beta \in \lambda$ or $\mu \circ (s^\alpha)^n \subseteq \lambda$ for some $n \in \mathbb{N}$.

Proposition 3.1. *Let $\mu \in F(S)$ and $A = S$. Then μ is a fuzzy primary subact of A iff μ is a fuzzy primary ideal of S.*

Proposition 3.2. *Let λ be a fuzzy primary subact of μ on A and $t \in Im(\lambda)$. If $\lambda_t \neq \mu_t$, then λ_t is a primary subact of μ_t.*

Example 3.3. Let $A = S = \mathbb{Z}$. We define:

$$\mu(x) = \begin{cases} 1 & if\ x \in 4\mathbb{Z} \\ \frac{1}{2} & if\ x \in 2\mathbb{Z} \setminus 4\mathbb{Z} \\ 0 & otherwise \end{cases} \quad , \quad \lambda(x) = \begin{cases} 1 & if\ x = 0 \\ \frac{1}{2} & if\ x \in 4\mathbb{Z} \setminus \{0\} \\ 0 & otherwise. \end{cases}$$

Then λ_t is a primary subact of μ_t for each $t \in (0, 1]$, but λ is not a fuzzy primary subact of μ, since if we put $r = \frac{2}{3}$, $t = \frac{1}{3}$, $y = 4$ and $x = 5$, then $x^t \circ y^r \in \lambda$ but $x^t \notin \lambda$ and $\mu \circ (y^s)^k \not\subseteq \lambda$ for all $k \in \mathbb{N}$.

Proposition 3.4. *Let B be a primary subact of A and $\alpha \in [0, 1)$. If λ is the fuzzy subset of A defined by $\lambda(x) = 1$ if $x \in B$ and $\lambda(x) = \alpha$ otherwise, for all $x \in A$. Then λ is a fuzzy primary subact of A.*

Proposition 3.5. *If μ is a fuzzy primary subact of A, then there exists $\alpha \in [0, 1)$ and a primary subact B of A such that $\mu(x) = 1$ for $x \in B$ and otherwise $\mu(x) = \alpha$.*

Let μ be a non-constant ideal in a commutative ring R. In 1990 Malik and Mordeson in "*Fuzzy prime ideals of a ring*" have proved that μ is prime iff $|Im(\mu)| = 2$ and μ_* be a prime ideal of R. The following theorem is a consequence of Propositions 3.4 and 3.5.

Theorem 3.6. *μ is a fuzzy prime ideal of S iff there exists $\alpha \in [0, 1)$ such that $\mu(x) = 1$ if $x \in \mu_*$ and $\mu(x) = \alpha$ if $x \notin \mu_*$, and μ_* is a prime ideal of S.*

4. Fuzzy P-Primary S-act

In this section first we describe fuzzy ideal $(\mu : \lambda)$ of S, next characterize fuzzy P-primary subacts of A.

Let $\lambda, \mu \in FS(A)$. Then $(\mu : \lambda)$ is the fuzzy subset of S defined by $(\mu : \lambda)(s) = \vee \{\alpha \in [0, 1] \mid \lambda \circ s^\alpha \subseteq \mu\}$ for all $s \in S$.

Proposition 4.1. *If $\lambda, \mu \in FS(A)$, then $(\mu : \lambda)$ is a fuzzy ideal of S.*

Lemma 4.2. *Let $\lambda, \mu, \omega \in FS(A)$ and $s^\alpha \in FP(S)$. Then*

(1) $\mu \circ (\lambda : \omega) \subseteq \lambda$.
(2) *If $\mu \subseteq \lambda$, then $(\mu : \omega) \subseteq (\lambda : \omega)$ and $(\omega : \lambda) \subseteq (\omega : \mu)$.*
(3) *If $\mu \subseteq \lambda$, then $(\lambda : \mu) = \chi_S$.*

Proposition 4.3. *If $\mu \in FS(A)$, then $(\mu : \chi_A)_* = (\mu_* : A)$.*

Theorem 4.4. *Let λ be a fuzzy primary subact of A. Then $\sqrt{(\lambda : \chi_A)}$ is a fuzzy ideal of S.*

Let μ be a fuzzy primary subact of A and $P = \sqrt{(\mu : \chi_A)}$. Then μ is called a *fuzzy P-primary subact* of A.

Proposition 4.5. *Let μ be a fuzzy P-primary subact of A, $a^\beta \in FP(A)$ and $s^\alpha \in FP(S)$. If $a^\beta \circ s^\alpha \in \mu$, then $a^\beta \in \mu$ or $s^\alpha \in P$.*

Proposition 4.6. *Let μ be a fuzzy P-primary subact of A, $\lambda \in FI(S)$ and ω be a fuzzy subact of A. If $\omega \circ \lambda \subseteq \mu$, then $\omega \subseteq \mu$ or $\lambda \subseteq P$.*

Theorem 4.7. *Let μ be a non-constant fuzzy subact of A and $P \in FI(S)$. Then μ is a fuzzy P-primary subact of A iff*

(1) *If $a^\beta \circ s^\alpha \in \mu$ and $a^\beta \notin \lambda$, then $s^\alpha \in P$ for all fuzzy points $a^\beta \in FP(A)$ and $s^\alpha \in FP(S)$ and*

(2) *If $s^\alpha \in P$, then $\exists n \in \mathbb{N}$ such that $\chi_{_A} \circ (s^\alpha)^n \subseteq \mu$.*

Theorem 4.8. *Let μ be a fuzzy P-Primary subact of A and $\lambda \in FS(A)$. If $(\mu : \lambda)$ is non-constant, then $(\mu : \lambda)$ is a fuzzy P-Primary ideal of A.*

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