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TETRAVALENT HALF-TRANSITIVE GRAPHS OF ORDER $2p^2$

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Abstract. A graph is *half-transitive* if its automorphism group acts transitively on its vertex set and edge set, but not on its arc set. Let p be a prime. Chao [On the classification of symmetric graphs with a prime number of vertices, Trans. Amer. Math. Soc. 158 (1971) 247-256] proved that there are no half-transitive graphs on p vertices. By Cheng and Oxley [On weakly symmetric graphs of order twice a prime, J. Combin. Theory B 42 (1987) 196-211], also there are no half-transitive graphs of order $2p$. In this paper an extension of the above results in the case of tetravalent graphs is given. It is proved that there are no tetravalent half-transitive graphs of order $2p^2$.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, unless otherwise specified, connected and undirected (but with an implicit orientation of the edges when appropriate). For a graph X we let $\mathrm{V}(X)$, $\mathrm{E}(X)$, $\mathrm{A}(X)$ and $\mathrm{Aut}(X)$ be the vertex set, edge set, arc set and the full automorphism group of X, respectively.

A graph X is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* if $\mathrm{Aut}(X)$ acts transitively on $\mathrm{V}(X)$, $\mathrm{E}(X)$ or $\mathrm{A}(X)$, respectively. A graph is said to be $\frac{1}{2}$-*transitive* or *half-transitive* provided that it is vertex-transitive and edge-transitive, but not arc-transitive. More generally, by a $\frac{1}{2}$-*transitive* action of a subgroup G of $\mathrm{Aut}(X)$ on a graph X we shall mean a vertex-transitive and edge-transitive, but not arc-transitive action of G on X. In this case we shall say that the graph X is $(G, \frac{1}{2})$-*transitive.*

The investigation of half-transitive graphs was initiated by Tutte and he proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In this paper, we show that there are no tetravalent half-transitive graphs of order $2p^2$.

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2. Preliminaries

For a finite group G, and a subset S of G such that $1_G \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{[g, sg] \mid g \in G, s \in S\}$. Given any element $g \in G$, we define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$ is a permutation group isomorphic to G, which is called the *right regular representation* of G. Actually, $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(Cay(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(Cay(G, S))$.

For any abelian group H, the map $h \mapsto h^{-1}$, $h \in H$, is an automorphism of H. In view of the proof of [4, Proposition 2.1], we have the following:

Proposition 2.1. *Let Cay(G,S) be a half-transitive graph. Then, there is no involution in S and no $\alpha \in$ $Aut(G, S)$ such that $s^\alpha = s^{-1}$ for any given $s \in S$.*

Next we quote a result from [1]

Proposition 2.2. *Every edge-transitive Cayley graph on an abelian group is also arc-transitive.*

Proposition 2.3. *There are no half-transitive graphs with fewer than 27 vertices.*

Proposition 2.4. *Let H be a subgroup of a group G. We have $C_G(H) \lhd N_G(H)$, and the factor group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.*

3. Main results

The following lemma is basic for our main result.

Lemma 3.1. *There are no tetravalent half-transitive Cayley graphs of order $2p^2$ for each prime p.*

By contradiction, let $X = Cay(G, S)$ be a tetravalent half-transitive Cayley graph on a group G of order $2p^2$ with respect to S. If X is not connected, then each component has order p, $2p$ or p^2. By [2, 3], there are no half-transitive graphs of order p or $2p$. Therefore each component has order p^2 and so each component is a Cayley graph of order p^2. By Proposition 2.2, there is no half-transitive Cayley graph on a group of order p^2, a contradiction. Hence, X is connected. By Proposition 2.3, one may let $p \geq 5$ and by Proposition 2.2, G is non-abelian. From the elementary group theory we know that up to isomorphism there are three non-abelian groups of order $2p^2$ defined by:
$G_1(p) = \langle a, b \mid a^{p^2} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$;
$G_2(p) = \langle a, b, c \mid a^p = b^p = c^2 = [a, b] = 1, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1}\rangle$;
$G_3(p) = \langle a, b, c \mid a^p = b^p = c^2 = 1, [a, b] = [a, c] = 1, c^{-1}bc = b^{-1}\rangle$.

Let G be a non-abelian group of order $2p^2$ and $S = \{x, y, x^{-1}, y^{-1}\}$ be a generating subset of G. If either of x or y has order 2, then by Proposition 2.1, X is half-transitive, a contradiction. Since the Sylow p-subgroup of G is a normal subgroup of G, any two elements of order p or p^2 cannot generate G. Thus we can suppose $o(x) = 2p$ and $o(y) = p$, $2p$ or p^2.

Now we prove that there exists an element of order p which is in the center of G. Note that $G = \langle x, y\rangle$. When $o(x) = 2p$ and $o(y) = p$ or p^2, it is easy to see that x^2 has order p and $x^2 \in Z(G)$. When $o(x) = 2p$ and $o(y) = 2p$, we have $|\langle x\rangle \cap \langle y\rangle| = 2$ or p. If $|\langle x\rangle \cap \langle y\rangle| = 2$, then the Sylow 2-subgroup of G is normal in G. Since the Sylow p-subgroup of G is also normal, G is abelian, a contradiction. Therefore $\langle x\rangle \cap \langle y\rangle$ has order p and $\langle x\rangle \cap \langle y\rangle \in Z(G)$, as required.

It is easily seen that only $G = G_3(p)$ has elements of order p which are in its center. Thus we can suppose that $G = G_3(p) = \langle a, b, c \mid a^p = b^p = c^2 = 1, [a, b] = [a, c] = 1, c^{-1}bc = b^{-1}\rangle$ and so $o(x) = 2p$ and $o(y) = p$ or $2p$.

It is easy to check that all the elements of order 2 are cb^j $(0 \le j < p)$. Thus we suppose that $x = cb^ja^i$ $(p \nmid i)$. Since a^i $(p \nmid i)$, b and cb^j satisfy the same relations as a, b and c, there is an automorphism σ of G such that $(a^i)^\sigma = a$, $b^\sigma = b$ and $(cb^j)^\sigma = c$. Hence we may suppose $x = ca$.

If $o(y) = p$, then we may suppose $y = a^ib$, by an argument similar to that above. Also with the same arguments as above, by considering Proposition 2.1, we may get a contradiction. Now the proof is completed.

The following is the main result of this paper.

Theorem 3.2. *Let p be a prime. Then there are no tetravalent half-transitive graphs of order $2p^2$.*

Let X be a tetravalent half-transitive graph of order $2p^2$. By Proposition 2.3, $p \ge 5$. Now X is connected because there are no half-transitive graphs of order p, $2p$ or p^2, by Propositions 2.2, 2.5, and [2 ,3]. By Lemma 3.1, X is not a Cayley graph. Let A=Aut(X). Then, A has no regular subgroups, that is, no subgroups acting regularly on V(X).

Under the natural action of A on V(X) $\times$ V(X), A has two orbits on the arc set of X, say A_1 and A_2. These are paired with each other, that is, $A_2 = \{(v, u)|(u, v) \in A_1\}$. Thus, now one can get $|A| = 2^mp^2$ for some integer m, implying that A is solvable. First we prove a claim.

Claim: A has a normal Sylow p-subgroup.

Suppose to the contrary that A has no normal Sylow p-subgroups. Let N be a minimal normal subgroup of A. Since $|A| = 2^mp^2$, $|N| = p$ or N has 2-power order.

First assume that $|N| = p$ and $T = \{x_1^N, x_2^N, ..., x_{2p}^N\}$ is the all orbits of N on V(X). Let X_N be the quotient graph of X corresponding to the orbits of N, with two orbits adjacent in X_N whenever there is an edge between those orbits in X. Then, $|V(X_N)| = 2p$. Also let K be the kernel of A acting on V(X_N). Clearly, A/K acts transitively on V(X_N) and E(X_N), respectively. Now if X_N has valency 3, then A/K acts transitively on A(X_N). It implies that $3 \mid |A|$, a contradiction. Hence X_N has valency 2 or 4. Suppose that X_N has valency 2. Then X_N is a cycle of length $2p$ and $|Aut(X_N)| = 4p$. Therefore $|A/K| \mid 4p$. Let $\mu \in$ V(X) and suppose that $K_\mu = 1$. It follows that $|K| = p$, $K = N$ and so A/N is a subgroup of Aut(X_N). Therefore $|A| \mid 4p^2$ and a Sylow p-subgroup of A, say P, is normal in A, a contradiction. Thus $K_\mu \ne 1$, which implies that $K_\mu \cong \mathbb{Z}_2$. Hence $|K| = 2p$. Since A/K is a subgroup of Aut(X_N), one has $|A/K| \mid 4p$. If $|A/K| \ne 4p$, then $P \lhd A$, a contradiction. Hence $|A/K| = 4p$ and so $|A| = 8p^2$. Thus $1 + np \mid 8$. It follows that $n = 1$ and $p = 7$. Let Q be a

Sylow 7-subgroup of K. Obviously Q is normal in K and hence Q is normal in A. Put $C = C_A(Q)$. By Proposition 2.4, A/C is isomorphic to a subgroup of $\mathrm{Aut}(Q) \cong \mathbb{Z}_6$. Hence $|A/C| \mid 2$, because $|A| = 8 \times 7^2$. It follows that $|C| = 4 \times 7^2$, or 8×7^2. For the first case $P \lhd C$ and so $P \lhd A$, a contradiction. For the latter case $A = C_A(P)$ and so $P \leq Z(A)$. Therefore $P \lhd A$, a contradiction. Thus X_N has valency 4, and we may get a same contradiction.

Now assume that N has order 2 power. Again we get a contradiction. Thus the claim is true, that is, A has a normal Sylow p-subgroup. Denote by N the unique normal Sylow p-subgroup of A. Let X_N be the quotient graph of X corresponding to the orbits of N, and K be the kernel of A acting $\mathrm{V}(X_N)$. The normality of N implies that all orbits of N either have length p or have length p^2. Assume that the orbits of N have length p. Thus p divides the order of N_α (for some $\alpha \in \mathrm{V}(X)$) and hence $|A_\alpha|$ is divisible by p. Therefore $|A_\alpha|$ has an element of order p, a contradiction. Now assume that the orbits of N have length p^2. Again we may get a contradiction. This contradiction completes our proof.

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