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SOME RESULTS ON WEAK ARMENDARIZ RINGS

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Abstract. According to Z. Liu and R. Zhao [5] a ring R is called *weak Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in nil(R)$ for all i, j. They have shown that, if R is a semicommutative ring, then the ring $R[x]$ and the ring $\frac{R[x]}{(x^n)}$, where (x^n) is the ideal generated by x^n, and n is a positive integer, are weak Armendariz. In this note we introduce weak Armendariz ideals which are a generalization of ideals have the weakly insertion of factors property (or simply weakly IFP) and investigate their properties.

1. Introduction

Throughout this paper R denotes an associative ring with identity. A ring R is called *semicommutative* if for any $a, b \in R, ab = 0$ implies $aRb = 0$. Rege and Chhawchharia introduced the notion of an Armendariz ring. A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0+a_1x+\cdots+a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. The name "Armendariz ring" was chosen because Armendariz (1974, Lemma 1) had noted that a *reduced* ring (i.e. $a^2 = 0$ implies $a = 0$) satisfies this condition. Some properties of Armendariz rings have been studied in Rege and Chhawchharia (1997), Armendariz (1974), Anderson and Camillo (1998), and Kim and Lee (2000).

We call an ideal I *weak Armendariz*, if whenever $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$, satisfy $f(x)g(x) \in I[x]$, then $a_ib_j \in \sqrt{I}$ for each i, j. Clearly if ideal $I = 0$ is weak Armendariz, then R is a weak Armendariz ring.

Recall that a one-sided ideal I of a ring R has *the insertion of factors property* (or simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. (H.E. Bell in 1973 introduced this notion for $I = 0$). Observe that every *completely semiprime* ideal (i.e., $a^2 \in I$ implies $a \in I$) of R has the IFP. If $I = 0$ has the IFP, then we say R has the IFP (or R is semicommutative). Li Liang et al. [4] introduced

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weakly semicommutative rings. A ring R is called *weakly semicommutative*, if for any $a, b \in R$, $ab = 0$ implies arb is a nilpotent element for each $r \in R$.

We say a one-sided ideal I of R has the *weakly IFP* if for each $a, b, r \in R, ab \in I$ implies $arb \in nil(I)$. Clearly, if ideal $I = 0$ has the weakly IFP, then R is a weakly semicommutative ring.

For a ring R, we denote by $nil(R)$ the set of all nilpotent elements of R and by $T_n(R)$ the n-by-n upper triangular matrix ring over R.

2. Main results

It is well-known that for a ring R and any positive integer $n \geq 2$,

$$\frac{R[x]}{(x^n)} \cong \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in R, i = 0, 1, \cdots, n-1 \right\},$$

where (x^n) is the ideal of $R[x]$ generated by x^n.

Lemma 2.1. *Let R be a ring and $n \geq 2$ a positive integer. Let $I_0, I_1, \cdots, I_{n-1}$ are ideals of R, such that $I_i \subseteq I_{i+1}, i = 0, 1, \cdots, n-2$. Then*

$$J = \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in I_i, i = 0, 1, \cdots, n \right\} \text{ is an ideal of } \frac{R[x]}{(x^n)}.$$

In Propositions 2.2, 2.5 and Theorem 2.3, I_0 and J are ideals that mentioned in Lemma 2.1.

Proposition 2.2. *Let* $A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} \in \frac{R[x]}{(x^n)}$, *such that $a_0^k \in I_0$ for some integer k. Then $A^{nk} \in J$.*

By using Proposition 2.2 we have the following theorem:

Theorem 2.3. *I_0 is weak Armendariz, if and only if J is weak Armendariz.*

Li Liang et al. [5 , Theorem 3.9] showed that, if R is a semicommutative ring, then the ring $\frac{R[x]}{(x^n)}$, for each positive integer n, is weak Armendariz. The following result is a generalization of Li Liang et al.'s result.

Corollary 2.4. *Let R be a ring. Then R is weak Armendariz, if and only if the ring $\frac{R[x]}{(x^n)}$, for each positive integer n, is weak Armendariz.*

Proposition 2.5. *I_0 has the weakly IFP, if and only if J has the weakly IFP.*

Clearly, if an ideal I has the IFP, then it has the weakly IFP. By the following example, we show that the converse is not true.

Example 2.6. *Let* $J = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a_i \in 2p\mathbb{Z} \right\}$ *be an ideal of* $\frac{\mathbb{Z}[x]}{(x^4)}$ *where* $2 \neq p$ *is a prime number and* $\mathbb{Z}$ *is the set of integers. We can show that* J *has not the IFP, but* J *has the weakly IFP, by Proposition 2.5.*

By a similar way as used in Example 2.6, we can construct numerous ideals of $\frac{\mathbb{Z}[x]}{(x^n)}$ such that have the weakly IFP, but have't the IFP, for $n \geq 2$.

Lemma 2.7. *Let* I *be an ideal of* R *and has the IFP. Then* $\sqrt{I}$ *is an ideal of* R *and has the IFP.*

Proposition 2.8. *Let* I *be an ideal of* R *and has the IFP. Then* I *and* $\sqrt{I}$ *are weak Armendariz.*

Corollary 2.9 (5, Corollary 3.4)**.** *Semicommutative rings are weak Armendariz.*

Proposition 2.10. *Let* $A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n(R)$ *such that* $a^k \in I$ *for some integer* k*. Then* $A^{nk} \in J$*.*

By using Proposition 2.12 we have the following theorem:

Theorem 2.11. I *is weak Armendariz if and only if* J *is weak Armendariz.*

Corollary 2.12 (5, Example 2.4)**.** *A ring* R *is weak Armendariz if and only if, for any positive integer* n*,* $R_n(R)$ *is weak Armendariz.*

Proposition 2.13. I *has the weakly IFP if and only if* J *has the weakly IFP.*

Corollary 2.14. *A ring* R *is weakly semicommutative, if and only if, for any positive integer* n*, the ring* $R_n(R)$ *is a weakly semicommutative ring.*

Proposition 2.15. *Let* R *be a ring and* I, J *be ideals of* R*. If* $I \subseteq \sqrt{J}$ *and* $\frac{I+J}{I}$ *is weak Armendariz, then* J *is weak Armendariz.*

Corollary 2.16 (5, Proposition 2.9)**.** *Let* R *be a ring and* I *an ideal of* R *such that* $\frac{R}{I}$ *is weak Armendariz. If* $I \subseteq nil(R)$*, then* R *is weak Armendariz.*

Theorem 2.17. J *is weak Armendariz if and only if all* I_{ii} *are weak Armendariz, for* $i = 1, \cdots, n$*.*

Corollary 2.18 (5, Proposition 2.2)**.** *A ring* R *is weak Armendariz if and only if* $T_n(R)$*, for any positive integer* n*, is weak Armendariz.*

Proposition 2.19. J *has the weakly IFP if and only if all* I_{ii} *has the weakly IFP, for* $i = 1, \cdots, n$*.*

Corollary 2.20 (4, Claim 2.1)**.** *A ring* R *is a weakly semicommutative ring if and only if, for any* n*, the* n*-by-*n *upper triangular matrix ring* $T_n(R)$ *is a weakly semicommutative ring.*

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