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ON SEMIHYPERGROUPS AND REGULAR RELATIONS

H. HEDAYATI

Department of Mathematics
Babol University of Technology
Babol, Iran h.hedayati@nit.ac.ir, hedayati143@yahoo.com

Abstract. We introduce two equivalence relations on semihypergroups. Also we discuss on some properties of these two relations and investigate on quotient semihypergroups via these relations.

1. Introduction and Preliminaries

The theory of algebraic hyperstructures which is a generalization of the concept of ordinary algebraic structures was first introduced by Marty [4]. Since then many researchers have worked on algebraic hyperstructures and developed it. A short review of this theory appears in [1]. A recent book [2] contains a wealth of applications. Via this book, Corsini and Leoreanu presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities.

A map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a *hyperoperation* or *join operation*. A *hypergroupoid* is a set H with together a (binary) hyperoperation $\circ$. A hypergroupoid $(H, \circ)$, which is associative, that is $x\circ(y\circ z) = (x\circ y)\circ z, \forall x, y, z \in H$, is called a *semihypergroup* (see [2]).

A semihypergroup $\mathcal{S}$ is said to have *zero element*, if there exists a unique element $e \in \mathcal{S}$ such that $ex = x = xe$, for all $x \in \mathcal{S}$. Note that we identify the singleton set, $\{x\}$ with x. Also a semihypergroup $\mathcal{S}$ is called *commutative*, if $xy = yx$, for all $x, y \in \mathcal{S}$.

In the sequel, by $\mathcal{S}$ we mean a semihypergroup, unless otherwise specified.

Definition 1.1. Let $(\mathcal{S}, .)$ be a semihypergroup. A nonempty subset $\mathcal{T}$ of $\mathcal{S}$ is called a *subsemihypergroup* of $\mathcal{S}$ if $(\mathcal{T}, .)$ is a semihypergroup

Let $\mathcal{S}$ be a semihypergroup and θ be an equivalence relation on $\mathcal{S}$. Naturally we can extend θ to the subsets of $\mathcal{S}$ denoted by $\overline{\theta}$ as follows.(see [1], [2])

For nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$. Define

$$\mathcal{A}\bar{\theta}\mathcal{B} \Longleftrightarrow \quad \forall a \in \mathcal{A} \;\; \exists b \in \mathcal{B}, \; a\theta b \quad \text{and} \quad \forall b \in \mathcal{B} \;\; \exists a \in \mathcal{A}, \; b\theta a,$$

where by $a\theta b$, we mean $(a, b) \in \theta$.

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An equivalence relation θ on $\mathcal{S}$ is said to be *regular* if for all $a, b, x \in \mathcal{S}$ we have

$$a\theta b \Longrightarrow (ax)\overline{\theta}(bx), \text{ and } (xa)\overline{\theta}(bx).$$

2. Regular relations

Definition 2.1. Let $\mathcal{S}$ be a semihypergroup. A subsemihypergroup $\mathcal{T}$ of $\mathcal{S}$ is called *invertible* if for all $x, y \in \mathcal{S}$ we have

$$x \in y\mathcal{T} \Longleftrightarrow y \in x\mathcal{T} \quad \text{and} \quad x \in \mathcal{T}y \Longleftrightarrow y \in \mathcal{T}x.$$

By $\mathcal{T} <_i \mathcal{S}$, we mean $\mathcal{T}$ is an invertible subsemihypergroup of $\mathcal{S}$.

If $\mathcal{S}$ is a semihypergroup, then it is clear that $\mathcal{S}$ itself is an invertible subsemihypergroup. Also if $\mathcal{S}$ has identity, then $\{e\}$ is an invertible subsemihypergroup.

Definition 2.2. Let $\mathcal{S}$ be a semihypergroup and $\{\mathcal{T}_j\}_{j=1}^n$ be a family of subsemihypergroups of $\mathcal{S}$, then the product of $\mathcal{T}_j$s is denoted by $\prod_{j=1}^{n} \mathcal{T}_j$ and is defined by $\prod_{j=1}^{n} \mathcal{T}_j = \{t \in \mathcal{S} | \ t \in \prod_{j=1}^{n} a_j, \exists a_j \in \mathcal{T}_j\}$. It is easy to see that $\prod_{j=1}^{n} \mathcal{T}_j$ is a subsemihypergroup.

Proposition 2.3. (i) Let $\mathcal{S}$ be a commutative semihypergroup and $\{\mathcal{T}_j\}_{j=1}^n$ be a family of invertible subsemihypergroups of $\mathcal{S}$, then $\prod_{j=1}^{n} \mathcal{T}_j$ is an invertible subsemihypergroup.

(ii) Let $\{\mathcal{S}_j\}_{j=1}^n$ be a family of semihypergroups and $\mathcal{T}_j <_i \mathcal{S}_j$ for all $1 \leq j \leq n$. Then $\mathcal{T}_1 \times \mathcal{T}_2 \times ... \times \mathcal{T}_n <_i \mathcal{S}_1 \times \mathcal{S}_2 \times ... \times \mathcal{S}_n$.

Proposition 2.4. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two semihypergroups and $f : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be an on-to homomorphism. If $\mathcal{T}$ is an invertible subsemihypergroup of $\mathcal{S}_1$, then $f(\mathcal{T})$ is an invertible subsemihypergroup of $\mathcal{S}_2$.

Definition 2.5. Let $\mathcal{S}$ be a commutative semihypergroup with identity and $\mathcal{T} < \mathcal{S}$. Define the relation $\lambda_{\mathcal{T}}$ on $\mathcal{S}$ as follows:

$x\lambda_{\mathcal{T}}y$ if there exist invertible subsemihypergroups $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{S}$ such that $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}$ and $x\mathcal{T}_1 \approx y\mathcal{T}_2$, where by $\mathcal{A} \approx \mathcal{B}$ we mean $\mathcal{A} \cap \mathcal{B} \neq \emptyset$.

Theorem 2.6. Let $\mathcal{S}$ be a commutative semihypergroup with identity and $\mathcal{T} < \mathcal{S}$. Then $\lambda_{\mathcal{T}}$ is a regular equivalence relation on $\mathcal{S}$.

Theorem 2.7. Let $(\mathcal{S}, .)$ be a commutative semihypergroup with identity and $\mathcal{S}/\lambda_{\mathcal{T}} = \{\lambda_{\mathcal{T}}(x) | \ x \in \mathcal{S}\}$ be the equivalence classes of $\mathcal{S}$ with respect to $\lambda_{\mathcal{T}}$. Then $(\mathcal{S}/\lambda_{\mathcal{T}}, \odot)$ is a commutative semihypergroup with $\lambda_{\mathcal{T}}(e)$ as an identity element, where $\odot$ is defined as follows:

$$\lambda_{\mathcal{T}}(x) \odot \lambda_{\mathcal{T}}(y) = \{\lambda_{\mathcal{T}}(z) | \ z \in \lambda_{\mathcal{T}}(x)\lambda_{\mathcal{T}}(y)\},$$

Definition 2.8. Let $\mathcal{S}$ be an commutative semihypergroup with identity and $\mathcal{T} < \mathcal{S}$. Define the relation $\rho_{\mathcal{T}}$ on $\mathcal{S}$ as follows:

$x\rho_{\mathcal{T}}y$ there exist invertible subsemihypergroups $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{S}$ such that $\mathcal{T}_1, \mathcal{T}_2 \subseteq I$ and $x\mathcal{T}_1 = y\mathcal{T}_2$.

Theorem 2.9. Let $\mathcal{S}$ be a commutative semihypergroup with identity, then $\rho_{\mathcal{T}} = \lambda_{\mathcal{T}}$.

Corollary 2.10. Let $\mathcal{S}$ be a commutative semihypergroup with identity. Then $\rho_{\mathcal{T}}$ is a regular equivalence relation on semihypergroup $\mathcal{S}$. Also $(\mathcal{S}/\rho_{\mathcal{T}}, \overline{\odot})$ is a commutative semihypergroup with identity, where $\overline{\odot}$ is defined as follows:

$$\rho_{\mathcal{T}}(x)\overline{\odot}\rho_{\mathcal{T}}(y) = \{\rho_{\mathcal{T}}(z)|\ z \in xy\},$$

Note 2.11. It is easy to verify that, if $\mathcal{T}_1 \subseteq \mathcal{T}$ is an invertible subsemihypergroup of $\mathcal{S}$, then for all $x \in \mathcal{S}$, we have $x\mathcal{T}_1 \subseteq \rho_{\mathcal{T}}(x) = \lambda_{\mathcal{T}}(x)$.

Lemma 2.12. Let $\mathcal{S}$ be a commutative semihypergroup with identity and $\mathcal{T}_1, \mathcal{T}_2 < \mathcal{S}$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Then $\rho_{\mathcal{T}}(x) \in \mathcal{T}_2/\rho_{\mathcal{T}}$ if and only if $x \in \mathcal{T}_2$.

Theorem 2.13. Let $\mathcal{S}$ be a commutative semihypergroup with identity and $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Then $\mathcal{T}_2 < \mathcal{S}$ if and only if $\mathcal{T}_2/\rho_{\mathcal{T}_1} < \mathcal{S}/\rho_{\mathcal{T}_1}$.

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