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VERTEX DECOMPOSABLE AND SHELLABLE COMPLETE T-PARTITE GRAPHS

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ABSTRACT. Let G be a simple undirected graph and let Δ_G be a simplicial complex whose faces correspend to the independent sets of G. we show that complete t-partite graph G is sequentially Cohen-Macaulay if and only if G is shellable. Also we show that complete t-partite graph G is vertex decomposable if and only if α_1 be arbitrary and $\alpha_i = 1$ for all $2 \leq i \leq t$.

1. Introduction

Let G be a simple undirected graph on the vertex set $\mathrm{V}(G) = \{x_1, \cdots, x_n\}$. By identifying the vertex v_i with the variable x_i in the polynomial ring $R = K[x_1, \cdots, x_n]$ over a field K, we can associate to G a quadratic square- free monomial ideal $I(G) =< x_i x_j | \{x_i, x_j\} \in \mathrm{E}(G) >$ where $E(G)$ is the edge set of G. The ideal $I(G)$ is called the edge ideal of G. Using the Stanley-Reisner correspendence , we can associate to G the simplicial complex Δ_G, where $I_{\Delta_G} = I(G)$. Notice that the faces of Δ_G are independent sets or stable sets of G. Thus F is a face of Δ_G if and only if there is no edge of G joining any two vertices of F.

We say that a graph G is complete t-partite graph if its vertex set V can be partitioned into disjoint subsets $v_1, \cdots, v_t$ such that $|v_i| = \alpha_i$ for all $1 \leq i \leq t$ and G contains every edge joining v_i and v_j for all $i \neq j$ and $1 \leq i, j \leq t$.

A simplicial complex Δ is recursively defined to be vertex decomposable if it is either a simplex, or else has some vertex v so that

i)Both $\Delta \setminus v$ and $link^v_\Delta$ are vertex decomposable and

ii)No face of $link^v_\Delta$ is a facet of $\Delta \setminus v$.

A simplicial complex Δ is called shellable if the facets (maximal faces) of Δ can be ordered $F_1, \ldots, F_s$ such that for all $1 \leqslant i < j \leqslant s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$. We call $F_1, \ldots, F_s$ a shelling of Δ when the facets have been ordered with respect to the definition

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of shellable. A graph G is called shellable, if the simplicial complex Δ_G is a shellable simplicial complex.
Let $R = k[x_1, \ldots, x_n]$. A graded R-module M is called sequentially Cohen-Macaulay (over k) if there exists a finite filtration of graded R-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

A graph G is said to be (sequentially)Cohen-Macaulay, if the ring $\frac{k[x_1,\cdots,x_n]}{I(G)}$ is a (sequentially)Cohen-Macaulay ring. We give a sufficient and necessary condition on a complete t-partite graph to be vertex decomposable. Note that all shellable graphs are sequentially Cohen-Macaulay but the converse isn't true in general.
our main result is to classify all sequentially Cohen-Macaulay complete t-partite graphs. Precisely we show that complete t-partite graph G is sequentially Cohen-Macaulay if and only if G is shellable.

2. Main results

Theorem 2.1. *Let G be a complete t-partite graph. G is vertex decomposable if and only if α_1 be arbitrary and $\alpha_i = 1$ for all $2 \le i \le t$.*

Theorem 2.2. *Let G be a complete t-partite graph. G is shellable if and only if α_1 be arbitrary and $\alpha_i = 1$ for all $2 \le i \le t$.*

Corollary 2.3. *Let G be a complete t-partite graph. G is shellable if and only if G is vertex decomposable.*

3. Alexander Duality

we show the following theorem.

Theorem 3.1. *Let G be a complete t-partite graph and Δ_G be independence complex of G. Then Alexander dual of Δ_G is sequentially Cohen-Macaulay.*

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