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EDGE-TRANSITIVE ELEMENTARY ABELIAN REGULAR COVER OF Q_3

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Abstract. A simple undirected graph is said to be semisymmetric if it is regular and edge-transitive but not vertex-transitive. Let $p \geq 11$ be a prime. In this paper, it is proved that, every cubic edge-transitive elementary abelian regular cover of Q_3 is vertex-transitive.

1. Introduction

In this paper we consider an undirected finite connected graph without loops or multiple edges. For a graph Γ, we denote by $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ its vertex set, edge set and automorphism group, respectively. For u, $v \in V(\Gamma)$, denote by uv the edge incident to u and v in Γ, and by $N_\Gamma(u)$ the *neighbourhood* of u in Γ, that is, the set of vertices adjacent to u in Γ. A graph $\widetilde{\Gamma}$ is called a *covering* of a graph Γ with projection $p : \widetilde{\Gamma} \to \Gamma$ if there is a surjection $p : V(\widetilde{\Gamma}) \to V(\Gamma)$ such that $p|_{N_{\widetilde{\Gamma}}(\tilde{v})} : N_{\widetilde{\Gamma}}(\tilde{v}) \to N_\Gamma(v)$ is a bijection for any vertex $v \in V(\Gamma)$ and $\tilde{v} \in p^{-1}(v)$. Let N be a subgroup of $\mathrm{Aut}(\Gamma)$ such that N is intransitive on $V(\Gamma)$. The quotient graph Γ/N induced by N is defined as the graph such that the set Σ of N-orbits in $V(\Gamma)$ is the vertex set of Γ/N and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $\{u, v\} \in E(\Gamma)$. A covering $\widetilde{\Gamma}$ of Γ with a projection p is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{\Gamma})$ such that graph Γ is isomorphic to the quotient graph $\widetilde{\Gamma}/K$, say by h, and the quotient map $\widetilde{\Gamma} \to \widetilde{\Gamma}/K$ is the composition ph of p and h (for the purpose of this paper, all functions are composed from left to right). If K is cyclic or elementary abelian then $\widetilde{\Gamma}$ is called a *cyclic* or an *elementary abelian covering* of Γ, and if $\widetilde{\Gamma}$ is connected K becomes the covering transformation group.

If a subgroup G of $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$, we say that Γ is *G-vertex-transitive*, *G-edge-transitive* and *G-arc-transitive*, respectively. In the special case, when $G = \mathrm{Aut}(\Gamma)$ we say that Γ is vertex-transitive,

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edge-transitive and arc-transitive (or *symmetric*), respectively. A regular G-edge-transitive but not G-vertex-transitive graph will be referred to as a *G-semisymmetric graph*. In particular, if G =Aut(Γ), then the graph Γ is said to be semisymmetric.

The study of semisymmetric graphs was initiated by Folkman [2]. It is given a classification of semisymmetric graphs of order $2pq$ in [1], where p and q are distinct primes.

Let Γ be a graph and K be a finite group. By a^{-1} we mean the reverse arc to an arc a. A *voltage assignment* (or, *K-voltage assignment*) of Γ is a function $\phi : A(\Gamma) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(\Gamma)$. The values of ϕ are called *voltages*, and K is the *voltage group*. The graph $\Gamma \times_\phi K$ derived from a voltage assignment $\phi : A(\Gamma) \to K$ has vertex set $V(\Gamma) \times K$ and edge set $E(\Gamma) \times K$, so that an edge (e, g) of $\Gamma \times K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(\Gamma)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $\Gamma \times_\phi K$ is a covering of Γ with the first coordinate projection $p : \Gamma \times_\phi K \to \Gamma$, which is called the *natural projection*. By defining $(u, g^{'})^g = (u, g^{'}g)$ for any $g \in K$ and $(u, g^{'}) \in V(\Gamma \times_\phi K)$, K becomes a subgroup of Aut($\Gamma \times_\phi K$) which acts semiregularly on $V(\Gamma \times_\phi K)$. Therefore, $\Gamma \times_\phi K$ can be viewed as a *K-covering*. For each $u \in V(\Gamma)$ and $uv \in E(\Gamma)$, the vertex set $\{(u, g) \mid g \in K\}$ is the fibre of u and the edge set $\{(u, g)(v, \phi(a)g) \mid g \in K\}$ is the fibre of uv, where $a = (u, v)$. Conversely, each regular covering $\widetilde{\Gamma}$ of Γ with a covering transformation group K can be derived from a K-voltage assignment. Given a spanning tree T of the graph Γ, a voltage assignment ϕ is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [5] showed that every regular covering $\widetilde{\Gamma}$ of a graph Γ can be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of Γ. It is clear that if ϕ is reduced, the derived graph $\Gamma \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltages group K.

Let $\widetilde{\Gamma}$ be a K-covering of Γ with a projection p. If $\alpha\in$ Aut(Γ) and $\tilde{\alpha}\in$ Aut($\widetilde{\Gamma}$) satisfy $\tilde{\alpha}p = p\alpha$, we call $\tilde{\alpha}$ a *lift* of α, and α the *projection* of $\tilde{\alpha}$. Concepts such as a lift of a subgroup of Aut(Γ) and the projection of a subgroup of $\widetilde{\Gamma}$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in Aut($\widetilde{\Gamma}$) and Aut(Γ) respectively. In particular, if the covering graph $\widetilde{\Gamma}$ is connected, then the covering transformation group K is the lift of the trivial group, that is K=$\{\tilde{\alpha}\in$ Aut($\widetilde{\Gamma}$): $p = \tilde{\alpha}p\}$. Clearly, if $\tilde{\alpha}$ is a lift of α, then $K\tilde{\alpha}$ are all the lifts of α.

2. Main results

Lemma 2.1. *Suppose that Γ is a connected semisymmetric cubic graph of order $8p^n$. Then Γ is a connected N-regular covering of Q_3 such that the subgroup of Aut(Q_3) generated by α and β lifts, where $|N| = p^n$*

Lemma 2.2. *Let $N \cong \mathbb{Z}_p^n$ and suppose that $\Gamma = Q_3 \times_\phi \mathbb{Z}_p^n$ is a connected $\mathbb{Z}_p^n$-regular covering of Q_3. If the subgroup of Aut(Q_3) generated by α and β can be lifted then Γ is symmetric.*

Theorem 2.3. *Let $p \geq 11$ be a prime. Then every cubic edge-transitive elementary abelian regular cover of Q_3 is vertex-transitive.*

References

[1] S. F. Du and M. Y. Xu, A classification of semisymmetric graphs of order $2pq$, *Com. in Algebra*, **28(6)** (2000), 2685-2715.
[2] J. Folkman, Regular line-symmetric graphs, *J. Combin. Theory*, **3** (1967), 215-232.
[3] A. Malnič, D. Marušič and C. Q. Wang, Cubic edge-transitive graphs of order $2p^3$, *Discrete Math.*, **274** (2004), 187-198.
[4] W. T. Tutte, *Connectivity in graphs*, Toronto University Press, 1966.
[5] J.L. Gross, T.W. Tucker, Generating all graph covering by permutation voltages assignment, Discrete Math. 18 (1977) 273-283.