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# EDGE-TRANSITIVE ELEMENTARY ABELIAN REGULAR COVER OF $Q_3$

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ABSTRACT. A simple undirected graph is said to be semisymmetric if it is regular and edge-transitive but not vertex-transitive. Let  $p \geq 11$ be a prime. In this paper, it is proved that, every cubic edge-transitive elementary abelian regular cover of  $Q_3$  is vertex-transitive.

# 1. INTRODUCTION

In this paper we consider an undirected finite connected graph without loops or multiple edges. For a graph  $\Gamma$ , we denote by  $V(\Gamma)$ ,  $E(\Gamma)$  and  $Aut(\Gamma)$  its vertex set, edge set and automorphism group, respectively. For  $u, v \in V(\Gamma)$ , denote by uv the edge incident to u and v in  $\Gamma$ , and by  $N_{\Gamma}(u)$  the neighbourhood of u in  $\Gamma$ , that is, the set of vertices adjacent to u in  $\Gamma$ . A graph  $\Gamma$  is called a covering of a graph  $\Gamma$  with projection  $p: \widetilde{\Gamma} \to \Gamma$  if there is a surjection  $p: V(\widetilde{\Gamma}) \to V(\Gamma)$  such that  $p|_{N_{\widetilde{\Gamma}}(\widetilde{v})} : N_{\widetilde{\Gamma}}(\widetilde{v}) \to N_{\Gamma}(v)$  is a bijection for any vertex  $v \in V(\Gamma)$  and  $\tilde{v} \in p^{-1}(v)$ . Let N be a subgroup of Aut( $\Gamma$ ) such that N is intransitive on  $V(\Gamma)$ . The quotient graph  $\Gamma/N$  induced by N is defined as the graph such that the set  $\Sigma$  of N-orbits in  $V(\Gamma)$  is the vertex set of  $\Gamma/N$  and  $B, C \in \Sigma$  are adjacent if and only if there exist  $u \in B$  and  $v \in C$  such that  $\{u, v\} \in E(\Gamma)$ . A covering  $\widetilde{\Gamma}$  of  $\Gamma$  with a projection p is said to be regular (or K-covering) if there is a semiregular subgroup K of the automorphism group Aut( $\overline{\Gamma}$ ) such that graph  $\Gamma$  is isomorphic to the quotient graph  $\overline{\Gamma}/K$ , say by h, and the quotient map  $\widetilde{\Gamma} \to \widetilde{\Gamma}/K$  is the composition ph of p and h (for the purpose of this paper, all functions are composed from left to right). If K is cyclic or elementary abelian then  $\Gamma$  is called a *cyclic* or an *elementary abelian* covering of  $\Gamma$ , and if  $\Gamma$  is connected K becomes the covering transformation group.

If a subgroup G of Aut( $\Gamma$ ) acts transitively on  $V(\Gamma)$ ,  $E(\Gamma)$  and  $A(\Gamma)$ , we say that  $\Gamma$  is *G*-vertex-transitive, *G*-edge-transitive and *G*-arc-transitive, respectively. In the special case, when  $G = \operatorname{Aut}(\Gamma)$  we say that  $\Gamma$  is vertex-transitive,

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edge-transitive and arc-transitive (or *symmetric*), respectively. A regular *G*-edge-transitive but not *G*-vertex-transitive graph will be referred to as a *G*-semisymmetric graph. In particular, if  $G = \operatorname{Aut}(\Gamma)$ , then the graph  $\Gamma$  is said to be semisymmetric.

The study of semisymmetric graphs was initiated by Folkman [2]. It is given a classification of semisymmetric graphs of order 2pq in [1], where p and q are distinct primes.

Let  $\Gamma$  be a graph and K be a finite group. By  $a^{-1}$  we mean the reverse arc to an arc a. A voltage assignment (or, K-voltage assignment) of  $\Gamma$  is a function  $\phi : A(\Gamma) \to K$  with the property that  $\phi(a^{-1}) = \phi(a)^{-1}$  for each arc  $a \in A(\Gamma)$ . The values of  $\phi$  are called voltages, and K is the voltage group. The graph  $\Gamma \times_{\phi} K$  derived from a voltage assignment  $\phi : A(\Gamma) \to K$  has vertex set  $V(\Gamma) \times K$  and edge set  $E(\Gamma) \times K$ , so that an edge (e, g) of  $\Gamma \times K$  joins a vertex (u, g) to  $(v, \phi(a)g)$  for  $a = (u, v) \in A(\Gamma)$  and  $g \in K$ , where e = uv.

Clearly, the derived graph  $\Gamma \times_{\phi} K$  is a covering of  $\Gamma$  with the first coordinate projection  $p: \Gamma \times_{\phi} K \to \Gamma$ , which is called the *natural projection*. By defining  $(u, g')^g = (u, g'g)$  for any  $g \in K$  and  $(u, g') \in V(\Gamma \times_{\phi} K)$ , K becomes a subgroup of  $\operatorname{Aut}(\Gamma \times_{\phi} K)$  which acts semiregularly on  $V(\Gamma \times_{\phi} K)$ . Therefore,  $\Gamma \times_{\phi} K$  can be viewed as a *K*-covering. For each  $u \in V(\Gamma)$  and  $uv \in E(\Gamma)$ , the vertex set  $\{(u,g) \mid g \in K\}$  is the fibre of u and the edge set  $\{(u,g)(v,\phi(a)g) \mid g \in K\}$  is the fibre of uv, where a = (u,v). Conversely, each regular covering  $\widetilde{\Gamma}$  of  $\Gamma$  with a covering transformation group K can be derived from a K-voltage assignment. Given a spanning tree T of the graph  $\Gamma$ , a voltage assignment  $\phi$  is said to be T-reduced if the voltages on the tree arcs are the identity. Gross and Tucker [5] showed that every regular covering  $\widetilde{\Gamma}$  of a graph  $\Gamma$  can be derived from a T-reduced voltage assignment  $\phi$  with respect to an arbitrary fixed spanning tree T of  $\Gamma$ . It is clear that if  $\phi$  is reduced, the derived graph  $\Gamma \times_{\phi} K$  is connected if and only if the voltages on the cotree arcs generate the voltages group K.

Let  $\Gamma$  be a K-covering of  $\Gamma$  with a projection p. If  $\alpha \in \operatorname{Aut}(\Gamma)$  and  $\tilde{\alpha} \in \operatorname{Aut}(\Gamma)$ satisfy  $\tilde{\alpha}p = p\alpha$ , we call  $\tilde{\alpha}$  a *lift* of  $\alpha$ , and  $\alpha$  the *projection* of  $\tilde{\alpha}$ . Concepts such as a lift of a subgroup of  $\operatorname{Aut}(\Gamma)$  and the projection of a subgroup of  $\tilde{\Gamma}$  are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in  $\operatorname{Aut}(\tilde{\Gamma})$  and  $\operatorname{Aut}(\Gamma)$  respectively. In particular, if the covering graph  $\tilde{\Gamma}$  is connected, then the covering transformation group K is the lift of the trivial group, that is  $K = \{\tilde{\alpha} \in \operatorname{Aut}(\tilde{\Gamma}): p = \tilde{\alpha}p\}$ . Clearly, if  $\tilde{\alpha}$  is a lift of  $\alpha$ , then  $K\tilde{\alpha}$  are all the lifts of  $\alpha$ .

#### 2. Main results

**Lemma 2.1.** Suppose that  $\Gamma$  is a connected semisymmetric cubic graph of order  $8p^n$ . Then  $\Gamma$  is a connected N-regular covering of  $Q_3$  such that the subgroup of  $Aut(Q_3)$  generated by  $\alpha$  and  $\beta$  lifts, where  $|N| = p^n$ 

**Lemma 2.2.** Let  $N \cong \mathbb{Z}_p^n$  and suppose that  $\Gamma = Q_3 \times_{\phi} \mathbb{Z}_p^n$  is a connected  $\mathbb{Z}_p^n$ -regular covering of  $Q_3$ . If the subgroup of  $Aut(Q_3)$  generated by  $\alpha$  and  $\beta$  can be lifted then  $\Gamma$  is symmetric.

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**Theorem 2.3.** Let  $p \ge 11$  be a prime. Then every cubic edge-transitive elementary abelian regular cover of  $Q_3$  is vertex-transitive.

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