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BAER INVARIANTS, VARIETAL COVERING GROUPS AND DIRECT LIMITS

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Abstract. In this talk, we study behaviors of Baer invariants and varietal covering groups with respect to direct limits. The main result has a useful application in order to extend some known structures of varietal covering groups for several famous products of finitely many to an arbitrary family of groups.

1. Introduction and Preliminaries

Historically, there have been several papers from the beginning of the twentieth century trying to find some structures for the well-known notion the covering group and its varietal generalization, the $\mathcal{V}$-covering group of some famous groups and products of groups, such as the direct product, the nilpotent and the regular product [1, 2, 3, 4].

In this talk, reviewing behaviors of direct limits on presentation of groups, verbal subgroups and Baer invariants, we show that the structure of $\mathcal{V}$-covering group commutes with the direct limit of a directed system of groups in some senses. Furthermore, we give an example showing that the hypothesis of being directed for the system of groups is an essential condition. Finally, as an application, we extend some of the previous formulas for the structure of $\mathcal{V}$-covering groups for finite direct product of groups to infinite ones.

Definition 1.1. Let $\mathcal{V}$ be a variety of groups and let G be a group. Then a $\mathcal{V}$-*covering group* of G (a varietal covering group of G with respect to $\mathcal{V}$) is a group G^* with a normal subgroup A such that $G^*/A \cong G$, $A \subseteq V(G^*) \cap V^*(G^*)$, and $A \cong \mathcal{V}M(G)$.

Definition 1.2. Let $\{G_i\}$ be a direct system of groups indexed by a partially ordered set I, which is also directed, that is, for every $i, j \in I$ there exists $k \in I$ such that $i, j \leq k$. For $i \leq j$, let there exists a homomorphism $\lambda_i^j : G_i \to G_j$ such that:

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(i) $\lambda_i^i : G_i \to G_i$ is the identity map of G_i, for all $i \in I$;
(ii) if $i \leq j \leq k$, then $\lambda_i^j \lambda_j^k = \lambda_i^k$, as the following commutative diagram:

$$\begin{array}{ccc} G_i & \xrightarrow{\lambda_i^j} & G_j \\ & {\scriptstyle\lambda_i^k}\searrow & \downarrow{\scriptstyle\lambda_j^k} \\ & & G_k. \end{array}$$

In this case, we call the system $\{G_i; \lambda_i^j, I\}$ a *directed system.* Now we define an equivalence relation on the disjoint union $\bigcup_{i\in I} G_i$, by: if $x \in G_i$ and $y \in G_j$, then

$$x \sim y \text{ if and only if } x\lambda_i^k = y\lambda_j^k \text{ for } k \geq i, j.$$

Let G denote the quotient set $\bigcup_{i\in I} G_i / \sim$ and use $\{x\}$ for the equivalence class of x. Also we define a multiplication on G as follows: if $\{x\}$, $\{y\}$ are elements of G, we choose $i, j \in I$ such that $x \in G_i$ and $y \in G_j$ then

$$\{x\}\{y\} = \{(x\lambda_i^k)(y\lambda_j^k)\}, \ for\ k \geq i, j.$$

Clearly this is a well-defined multiplication, which makes G into a group and it is called the *direct limit* of the directed system $\{G_i; \lambda_i^j, I\}$. It will be denoted by

$$\underrightarrow{\lim}\, G_i = \bigcup_{i\in I} G_i / \sim = G.$$

2. Main Results

In order to deal with $\mathcal{V}$-covering groups of a group G, it is useful to know more relationship between the Baer invariant $\mathcal{V}M(G)$ and the $\mathcal{V}$-covering groups of G. In this aspect, to prove our main theorem, first of all we need to point the following notes which are the generalization of some parts of an important theorem of Schur (see [1] Theorem 2.4.6 or [3]).

Theorem 2.1. *Let $\mathcal{V}$ be a variety of groups and G be a group with a free presentation $1 \to R \to F \to G \to 1$. If S is a normal subgroup of F such that*

$$\frac{R}{[RV^*F]} \cong \frac{R \cap V(F)}{[RV^*F]} \times \frac{S}{[RV^*F]},$$

then $G^* = F/S$ is a $\mathcal{V}$-covering group of G.

Theorem 2.2. *Let $\mathcal{V}$ be a Schur-Baer variety and G be a finite group with a free presentation $1 \to R \to F \to G \to 1$. If G^* is $\mathcal{V}$-covering group of G, then there exists a normal subgroup S of F such that*

$$\frac{R}{[RV^*F]} \cong \frac{R \cap V(F)}{[RV^*F]} \times \frac{S}{[RV^*F]} \ ,$$

and so $G^* \cong F/S$.

Now, in order to state and prove the next result we need to explain the concept of an induced directed system of $\mathcal{V}$-covering groups which we use in the main theorem. Let $\mathcal{V}$ be a Schur-Baer variety, and let $\{G_i; \lambda_i^j, I\}$ be a directed system of finite groups. suppose that G_i^* is a $\mathcal{V}$-covering group for G_i, for all

$i \in I$. Now if we consider the sequence

$$1 \to R_i \to F_i \to G_i \to 1$$

as a free presentation, then using Theorem 2.2 there exists a normal subgroup S_i of F_i in such a way that $G_i^* = F_i/S_i$ and specially satisfies the following relation:

$$\frac{R_i}{[R_iV^*F_i]} \cong \frac{R_i \cap V(F_i)}{[R_iV^*F_i]} \times \frac{S_i}{[R_iV^*F_i]} .$$

By these notations, for any $i \le j$ in I, there exists an induced homomorphism $\widehat{\lambda}_i^j$ commutes the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R_i & \longrightarrow & F_i & \longrightarrow & G_i & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle \widehat{\lambda}_i^j} & & \downarrow{\scriptstyle \lambda_i^j} & & \\ 1 & \longrightarrow & R_j & \longrightarrow & F_j & \longrightarrow & G_j & \longrightarrow & 1. \end{array}$$

The commutativity of this diagram implies that the homomorphism $\widehat{\lambda}_i^j$ maps R_i into R_j and so $\widehat{\lambda}_i^j(R_i \cap V(F_i)) \subseteq R_j \cap V(F_j)$. Hence if $\widehat{\lambda}_i^j$ maps S_i into S_j, we will have the following induced homomorphism, for any $i \le j$:

$$\widetilde{\lambda}_i^j : G_i^* = \frac{F_i}{S_i} \longrightarrow G_j^* = \frac{F_j}{S_j},$$

which forms the directed system $\{G_i^*; \widetilde{\lambda}_i^j, I\}$, called an *induced directed system of covering groups.*

Remark 2.3. Note that in general, any family of covering groups of a directed system of groups is not necessarily an induced one. For example we consider the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ with two non-isomorphic covering groups D_8 and Q_8. So it takes the trivial directed system $\{G_i; \lambda_i^j, \mathbf{N}\}$ which $G_i = \mathbf{Z}_2 \times \mathbf{Z}_2$ and λ_i^j to be identity, for any $i, j \in \mathbf{N}$. Now if we take the family of covering groups $\{G_i^*;\ i \in \mathbf{N}\}$ such that $G_{2i}^* = D_8$ and $G_{2i+1}^* = Q_8$, then $\{G_i^*;\ i \in I\}$ does not form an induced directed system.

Theorem 2.4. *Suppose that $\mathcal{V}$ is a Schur-Baer variety. If $\{G_i; \lambda_i^j, I\}$ is a directed system of finite groups with an induced system of $\mathcal{V}$-covering groups $\{G_i^*; \widetilde{\lambda}_i^j, I\}$, as we mentioned above, then the group $G^* = \varinjlim G_i^*$ is a $\mathcal{V}$-covering group for $G = \varinjlim G_i$.*

Example 2.5. If we omit the condition of being directed, then the free product of any two groups A, B as a particular direct limit of groups whose its index set is not directed, should have a covering group with the structure $A^* * B^*$, where A^* and B^* are covering groups of A and B, respectively. But this is a contradiction, when we choose A with nontrivial Schur multiplier. Since in this case, if $A^* * B^*$ is a covering group of $A * B$, then by Definition 1.1, we will have

$$M(A * B) \cong N \text{ with } N \subseteq Z(A^* * B^*) \cap (A^* * B^*)' = 1.$$

But using a result of Miller (see [1]), we have $M(A * B) \cong M(A) \times M(B) \neq 1$, which is a contradiction.

3. Applications

Corollary 3.1. *Let G be a torsion nilpotent group with its all Sylow subgroups S_i, for $i \in I$. Suppose S_i^* is a covering group of S_i, for all $i \in I$, then the group $G^* = \prod_{i\in I}^{\times} S_i^*$ is a covering group for G.*

Corollary 3.2. *Let G be a torsion nilpotent group with its all Sylow subgroups P_i, for $i \in I$. Suppose P_i^* is an $\mathcal{N}_c$-covering group of P_i, for all $i \in I$. Then the group $G^* = \prod_{i\in I}^{\times} P_i^*$ is an $\mathcal{N}_c$-covering group for G.*

Corollary 3.3. *Let $\{A_i\}_{i\in I}$ be an arbitrary family of finite groups and suppose that A_i^* is a covering group of A_i, for any $i \in I$. Then the second nilpotent product of A_i's, is a covering group of $\prod_{i\in I}^{\times} A_i$.*

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