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THE BAER CRITERION FOR ACTS OVER CLIFFORD SEMIGROUPS

GH. MOGHADDASI

Sabzevar Teacher Training University
Sabzevar, Iran
moghaddasimath@yahoo.com

Abstract. It is known that although the Baer Criterion for injectivity holds for modules over rings with unit, it is not true for acts over an arbitrary monoid. Recently, the present author, together with M. Ebrahimi and M. Mahmoudi, published a paper (Comm. Algebra 35 (2007), 3912-3918) giving for the acts over some classes of semigroups, the Baer Criterion is true. In this paper we find another classe of monoids such that for acts over them the Baer Criterion hold.

1. Introduction

Recall that a (right) S-*act* is a set A together with a function $\alpha : A \times S \to A$, called the *action* of S (or the S-action) on A, such that for $x \in A$ and $s, t \in S$ (denoting $\alpha(x, s)$ by xs), $x(st) = (xs)t$. If S were a monoid with the identity element e, we also required the condition $xe = x$. A function $f : A \to B$ between S-acts A, B is called *an* S-*act map* (simply an *act map*) or a *homomorphism* if for each $x \in A$, $s \in S$, $f(xs) = f(x)s$.

Since the identity maps and the composite of two act maps are act maps, we have the category, **Act-S**, of all (right) S-acts and act maps between them.

The class of S-acts is an equational class, and so the category **Act-S** is complete (has all products and equalizers) and cocomplete (has all coproducts and coequalizers). In fact, limits and colimits in this category are computed as in sets equipped with a natural action. Also, monomorphisms of this category are exactly one-one act maps.

An S-act B containing (an isomorphic copy of) an S-act A as a subact is called an *extension* of A. The S-act A is said to be a *retract* of its extension B if there exists a homomorphism $f : B \to A$ such that $f\restriction_A = id_A$, in which case f is said to be a *retraction.* A is called *absolute retract* if it is a retract of each of its extensions.

An S-act A is said to be *injective* if for every monomorphism $h : B \to C$ and every homomorphism $f : B \to A$ there exists a homomorphism $g : C \to A$ such that $gh = f$.

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Also, recall that an element a of an S-act A is called a *fixed* or a *zero* element if $as = a$ for all $s \in S$.

Notice that injective acts over monoids have necessarily a zero element (see [4]) and the same is proved similarly for acts over semigroups. We recall the following theorem from [2].

Lemma 1.1. *For any semigroup S, in the category of S-acts, pushouts preserve monomorphisms.*

One can now use the above lemma and the results of Banaschewki [1] to get the followings:

Theorem 1.2. *Let S be an arbitrary semigroup. For an act A over S, we have:*

(1) *A is injective if and only if A is absolute retract, if and only if A has no proper essential extensions.*

(2) *B is an injective hull of A if and only if B is a maximal essential extension of A, if and only if B is a minimal injective extension of A.*

Definition 1.3. *An S-act A is said to be weakly injective if each act map $f : I \to A$ can be extended to an act map from S to A, where I is an arbitrary right ideal (a subset of S which is closed under right multiplication by elements of S).*

The condition that weakly injectivity is equivalent to injectivity is known as the *Baer Criterion* for injectivity. However, although this condition is true for injectivity of R-modules for any ring R with unit, it is not true for injectivity of M-acts, for an arbitrary monoid M. For example consider $(\mathbb{N}, max)$ (see [4]).

The notion of completeness, telling that each "Cauchy sequence converges", have first been applied to projection algebras (see [3]), which are acts over the monoid $(\mathbb{N}^\infty, min)$.

Definition 1.4. A *Cauchy sequence* over an S-act A is a family $(a_s)_{s\in S}$ of elements of A with $a_s t = a_{st}$ for all $s, t \in S$.

By a *limit* of a Cauchy sequence $(a_s)_{s\in S}$ over A in some extension B of A we mean an element $b \in B$ such that $bs = a_s$ for all $s \in S$.

It is easily proved that

Lemma 1.5. *A sequence $(a_s)_{s\in S}$ over an S-act A has a limit in some extension B of A if and only if it is a Cauchy sequence.*

Notice that limits of a Cauchy sequence over an act A in some extension B of A are not necessarily unique, unless B is *separated*, in the sense that if $bs = b's$ for all $s \in S$ then $b = b'$.

Definition 1.6. An S-act A is said to be *complete* if any Cauchy sequence over A has a limit in A.

Theorem 1.7. *Let S be a semigroup. An S-act A is complete if and only if for every homomorphism $f : S \to A$ there exist $a \in A$ such that $f = \lambda_a$ that is $f(s) = as$ for all $s \in S$.*

2. Baer Criterion

First according to [5], recall that a Clifford semigroup S is the form $S = \bigcup_{\alpha \in Y} G_\alpha$, where $\{G_\alpha\}_{\alpha \in Y}$ is a family of disjoint groups indexed by a semilattice Y. Suppose that for all $\alpha \geq \beta$ in Y there exists a group homomorphism $f_{\alpha,\beta} : G_\alpha \to G_\beta$ such that

(1) $f_{\alpha,\alpha} = id_{G_\alpha}$ for all $\alpha \in Y$,

(2) for all $\alpha \geq \beta \geq \gamma$ in Y, $f_{\beta,\gamma} f_{\alpha,\beta} = f_{\alpha,\gamma}$.

Define a multiplication in $S = \bigcup_{\alpha \in Y} G_\alpha$, by

$$xy = f_{\alpha,\alpha\wedge\beta}(x) f_{\beta,\alpha\wedge\beta}(y)$$

for $x \in G_\alpha, y \in G_\beta$. Then S is said to be a strong semilattice Y of groups $G_\alpha, \alpha \in Y$, with defining homomorphisms $f_{\alpha,\beta} \alpha \geq \beta$; S defined in this manner is a semigroup, indeed a Clifford semigroup may be constructed in this way (see[4]).

Let e_α be the identity in G_α for any $\alpha \in Y$. Then $E = \{e_\alpha | \ \alpha \in Y\}$ is the set of idempotents that are central in S. If $B \subseteq S$ then $B = \{x_\beta \in G_\beta | \ G_\beta \cap B \neq \emptyset, \beta \in Y\}$. Set

$$Y_\beta = \{\beta \in Y | \ \exists x_\beta \in B\}$$

which is a subset of Y.

By $\beta \leq H$, where $H \subseteq Y$, we mean that $\beta \in Y$ and $\beta \leq \alpha$ for some $\alpha \in H$. In particular if $H = \{\alpha\}$ we write $\beta \leq \alpha$.

It is proved in [5] Corrolary 2.2:

Theorem 2.1. *Let Y be a semilattice, $S = \bigcup_{\alpha \in Y} G_\alpha$ a strong semilattice of groups G_α. Then I is a principal ideal of S if and only if $I = \bigcup_{\beta \leq \alpha} G_\alpha = e_\alpha S$ for some $\alpha \in Y$.*

Theorem 2.2. *Let S be a Clifford semigroup each of whose proper non empty ideal is principal. Then every S-act A with at least one zero element is complete if and only if it is injective.*

Corollary 2.3. *Let S be a Clifford semigroup each of whose proper non empty ideal is principal. Then the Baer Criterion holds for S^e-acts with at least one zero element.*

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