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ON RIGHT PRINCIPALLY PROJECTIVE SKEW POWER SERIES RINGS

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Abstract. In this paper we give a generalization of a result of J.A. Fraser and W.K. Nicholson [2]. Let R be a ring and α be a weakly rigid automorphism of R. If R is skew power series Armendariz, then $R[[x;\alpha]]$ is right principally projective if and only if R is right principally projective and any countable family of idempotents in R has a generalized join when all left semicentral idempotents are central.

1. Introduction

We want to construct polynomials over a (not necessarily commutative) ring R in one variable Y which needs not commute with elements of R. Further we want a unique representation for each non-zero polynomial of the form $\sum_{i=0}^{p} r_i Y^i$, where $r_o, \cdots, r_p \in R$ and $r_p \neq 0$, i.e. we want the polynomial ring to be a free left R-module with basis $\{I, Y, Y^2, \cdots\}$. Degrees of polynomials, defined as

$$deg\left(\sum_{i=0}^{p} r_i Y^i\right) := p,$$

(where $r_p \neq 0$) shall be respected by multiplication, in the following sense:

$$deg(fg) \leq deg(f) + deg(g).$$

This means in particular that for $r \in R$ we want to have

$$Yr = \alpha(r)Y + \delta(r)$$

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for some mappings $\alpha,\ \delta : R \to R$. Consequently a and δ need to have some special properties: By ring properties we conclude for $r, r' \in R$

$$\alpha(r+r')Y + \delta(r+r') = Y(r+r') = Yr + Yr' = (\alpha(r)+\alpha(r'))Y + \delta(r) + \delta(r')$$

and

$$\begin{aligned} \alpha(rr')Y + \delta(rr') &= Y(rr') = (Yr)r' = (\alpha(r)Yr' + \delta(r)r' \\ &= \alpha(r)Yr' + \delta(r)r' = \alpha(r)(\alpha(r')Y + \delta(r)) + \delta(r)r' \\ &= \alpha(r)\alpha(r')Y + \alpha(r)\delta(r') + \delta(r)r' \end{aligned}$$

which, using module basis properties, implies

$$\alpha(r+r') = \alpha(r) + \alpha(r)$$

$$\alpha(rr') = \alpha(r)\alpha(r')$$

(i.e. α is an endomorphism of R) and

$$\delta(r+r') = \delta(r) + \delta(r')$$

and

$$\delta(rr') = \alpha(r)\delta(r') + \delta(r)r'$$

(i.e. δ is an αderivation as defined below). It turns out, that these properties are not only necessary, but already sufficient for the existence of such a ring.

Throughout $\alpha : R \to R$ is an automorphism and $C(R)$ the center of R. We denote $S = R[[x;\alpha]]$ the skew power series ring, whose elements are power series of the form $\Sigma_{i=0}^{\infty} r_i x^i$ with coefficients $r_i \in R$, where the addition is defined as usual and the multiplication subject to the condition $xb = \alpha(b)x$, for any $b \in R$.

In [5] Rickart studied C^*-algebras with the property that every right annihilator of any element is generated by a projection (i.e., p is a projection if $p = p^2 = p^*$ where $*$ is the involution on the algebra). A ring satisfying a generalization of Rickarts condition (i.e., every right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right PP ring. A ring R is called a right (resp. left) PP ring if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of an element of R is generated (as a right (resp. left) ideal) by an idempotent of R). R is called a PP ring (also called a Rickart ring if it is both right and left PP. There is a right p.p.-ring which is not right p.q.-Baer.

In 1974, Armendariz showed that a reduced ring R is Baer if and only if $R[x]$ is Baer [1, Theorem B]. Armendariz also provided an example to show that the reduced condition is not superfluous.

A ring R is called *Armendariz* if whenever two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$ we have $a_i b_j = 0$ for every i, j.

Fraser and Nicholson in [2] showed that $R[[x]]$ is a reduced p.p.-ring if and only if R is a reduced p.p.-ring and any countable family of idempotents of R has a least upper bound in $I(R)$, the set of all idempotents.

Z. Liu in [4, Theorem 3], showed that: If R is a ring such that all left semicentral idempotents are central, then $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and any countable family of idempotents in R has a generalized join in $I(R)$.

A monomorphism α is said to be rigid if for each $a \in R$, $a\alpha(a) = 0$ implies that $a = 0$. According [3] a monomorphism α is said to be weakly rigid if for each $a, b \in R$, $ab = 0$ implies that $a\alpha(b) = \alpha(b)a = 0$. By [3, Proposition 3], every rigid endomorphism is weakly rigid. By [3] there are examples of weakly rigid endomorphism which are not rigid.

2. Skew power series Armendariz rings

Motivated by results in Armendariz [1], we investigate a generalization of α-rigid rings and introduce skew power series versions of the Armendariz rings:

Definition 2.1. For a ring R and an automorphism $\alpha : R \to R$, we say R is skew power series Armendariz, if for each $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$, $f(x)g(x) = 0$ if and only if $a_i b_j = 0$ for all i, j.

We first note that there is an example of a non reduced regular ring (hence p.p) that is neither right nor left p.q.-Baer.

Lemma 2.2. Let R be a weakly rigid ring. Then we have the following:
(i) If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$ for each positive integer n.
(ii) If $a\alpha^k(b) = 0$ for some positive integer k, then $ab = 0$.

3. Principally Projective Skew Power Series Rings

In this section, we give a necessary and sufficient condition for some rings under which the ring $R[[x;\alpha]]$ is right p.p.

Proposition 3.1. Let R be weakly rigid ring and S the skew power series ring $R[[x;\alpha]]$. Then the following statements are equivalent:
(1) R is skew power series Armendariz ;
(2) $\varphi : rAnn_R(2^R) \to rAnn_S(2^S); A \to AS$ is bijective;
(3) $\psi : \ell Ann_R(2^R) \to \ell Ann_S(2^S); B \to SB$ is bijective.

A ring is called *abelian* if every idempotent in it is central.

Proposition 3.2. Every skew power series Armendariz ring is abelian.
Definition 3.3. (Z. Liu, [4]). Let $\{e_0, e_1, \cdots\}$ be a countable family of idempotents of R. We say $\{e_0, e_1, \cdots\}$ has a join in $I(R)$ if there exists an idempotent $e \in I(R)$ such that

1. $e_i(1 - e) = 0$, and
2. If $f \in I(R)$ is such that $e_i(1 - f) = 0$, then $e(1 - f) = 0$.

Theorem 3.4. Let R be a weakly rigid skew power series Armendariz ring. Then $R[[x;\alpha]]$ is right p.p. if and only if R is right p.p. and any countable family of idempotents in R has a join in $I(R)$.

Corollary 3.5.(J.A. Fraser, W.K. Nicholson [2]) Let R be a reduced power series ring. Then $R[[x]]$ is right p.p. if and only if R is right p.p. and any countable family of idempotents in R has a join in $I(R)$.

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