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# MODULES WITH FINITELY MANY PRIME SUBMODULES

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Abstract. Let $M$ be a unitary module over a commutative ring $R$. We say that $M$ has $FMP$ property if every prime submodule of $M$ contains a finitely many prime submodules of $M$. In [4] we have given a characterization of free modules with this property. The main aim of this note is to study some properties of rings and modules with $FMP$ property. Also we will generalize Cohen's theorem for the Noetherian rings to the modules with $FMP$ property.

## 1. Introduction

Throughout this note, all rings are commutative with identity and all modules are unitary. For a submodule $N$ of an $R$-module $M$, the set $\{r \in R | \, rM \subseteq N\}$ is denoted by $(N : M)$ and is called colon of $N$. If $N$ is a proper submodule of $M$ and $rm \in N$, for some $r \in R$ and $m \in M$ implies either $m \in N$ or $r \in (N : M)$, then $N$ is said to be a prime submodule of $M$. In [2, Corollary 1.4], it is proved that:

**Corollary.** If $R$ is a Noetherian domain and $F$ is a free $R$-module such that every prime submodule of $F$ contains only finitely many prime submodules, then for any primary submodule $Q$ of $F$, rad $Q$ is prime.

This is a motivation for studying the modules $M$ for which every prime submodule of $M$ contains only finitely many prime submodules of $M$. In [4] we have given a characterization of free modules with this property.

**Definition 1.1.** We say that an $R$-module $M$ has $FMP$ property, if every prime submodule of $M$ contains only finitely many prime submodules of $M$. Also it is said that a ring $R$ has $FMP$ property if $R$ has $FMP$ property as an $R$-module.

For example every one-dimensional domain has $FMP$ property, since every nonzero prime ideal is maximal.

The aim of this note is to study the rings and modules which have $FMP$ property.

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## 2. Main results

### 2.1. Modules With Finitely Many Prime Submodules.

**Proposition 2.1.** *A finitely generated module $M$ has $FMP$ property if and only if every maximal submodule of $M$ contains only finitely many prime submodules of $M$.*

**Proposition 2.2.** *Let $F$ be a free (or a faithfully flat or a finitely generated faithful multiplication) $R$-module. If $F$ has $FMP$ property, then $R$ has $FMP$ property.*

**Remark 2.3.** Note that the converse of this proposition is true for finitely generated faithful multiplication modules, since every prime submodule of a finitely generated faithful multiplication module is of the form $PM$ for some prime ideal $P$ of $R$. In general, the converse of the Proposition 2.2, is not correct, although we have a particular case for the converse of it:

The set of all $P$-prime submodules of an $R$-module $M$ is denoted by $Spec_P\, M$. Also the set of all prime submodules of $M$ is denoted by $Spec\, M$. The $R$-module $M$ is said to be *Zariski-bounded* if there exists a positive integer $n$ such that for every prime ideal $P$ of $R$, $\mid Spec_P\, M \mid < n$. It is easy to see that if $M$ is a *Zariski-bounded* module and $R$ has $FMP$ property, then $M$ also has $FMP$ property. So the converse of Proposition 2.2 is true for *Zariski-bounded* modules. In general, if for each prime ideal $P$ of $R$, $Spec_P\, M$ is a finite set and $R$ has $FMP$ property, then $M$ has $FMP$ property, as an $R$-module.

The prime submodule dimension of an $R$-module $M$ is defined by

$$dim\, M = sup_k \{N_0 \subset N_1 \subset \ldots \subset N_k \,|\, \text{each}\, N_i \text{ is a prime submodule of}\, M\}.$$

**Lemma 2.4.** *If $M$ has $FMP$ property, then $dim\, M$ is finite.*

An $R$-module $M$ is said to be a *serial module* if every two submodules of $M$ are comparable. It is easy to see that the converse of the above Lemma is true for serial modules.

**Proposition 2.5.** *Let $M$ be an $R$ module. The following are equivalent:*

(i) *$M$ has $FMP$ property;*
(ii) *for each prime ideal $P$ of $R$, $M_P$ has $FMP$ property;*
(iii) *for each maximal ideal $P$ of $R$, $M_P$ has $FMP$ property.*

A ring $R$ is said to be an *arithmetical ring*, if for all ideal $I, J$ and $K$ of $R$ we have, $I + (J \cap K) = (I + J) \cap (I + K)$. It is easy to see that, Prüfer domains and Dedekind domains are arithmetical.

**Theorem 2.6.** *Every finite dimensional arithmetical ring has $FMP$ property.*

**Theorem 2.7.** *Let $R$ be a Noetherian ring. If there exists a free (or a faithfully flat or a finitely generated faithful multiplication) $R$-module $F$ with $FMP$ property, then $dim\, R = 1$.*

**Corollary 2.8.** *If $R$ is a Noetherian ring with $FMP$ property, then dim $R = 1$.*

**Theorem 2.9.** *Let $F$ be a free $R$-module and $2 < rank_R\ F < \infty$. Then, $F$ has $FMP$ property if and only if $\frac{R}{P}$ is a finite field, for each prime ideal $P$ of $R$.*

**Theorem 2.10.** *Let $M$ be a module over a zero-dimensional ring $R$. Then $M$ has $FMP$ property if and only if for each prime submodule $N$ of $M$, $\frac{M}{N}$ has $FMP$ property as an $R$-module.*

2.2. **On Minimal Prime Submodules of Modules with $FMP$ Property.** In [1], Lu proves the extension of Cohen's theorem to modules, namely a finitely generated module $M$ is Noetherian if and only if every prime submodule of $M$ is finitely generated. In this section we will prove a generalization of Cohen's theorem for modules with $FMP$ property. First we have a few useful results:

**Theorem 2.11.** *Let $M$ be a serial module. The following statements are equivalent:*

(i) *$M$ has $FMP$ property;*
(ii) *$M$ satisfies $A.C.C$ on prime submodules;*
(iii) *$M$ satisfies $A.C.C$ on radical submodules.*

**Theorem 2.12.** *Let $M$ be a serial module. If $M$ satisfies $FMP$ property, then each radical submodule is a radical of a finitely generated submodule.*

**Corollary 2.13.** *Let $M$ be an $R$-module. If $M$ satisfies $A.C.C$ on radical submodules, then each prime submodule is a radical of a finitely generated submodule.*

**Corollary 2.14.** *Let $M$ be a serial module. If $M$ satisfies $FMP$ property, then every prime submodule is minimal over a finitely generated submodule.*

The following theorem is particularly a converse of Corollary 2.13.

**Theorem 2.15.** *Let $M$ be an $R$-module. If each prime submodule is a radical of a finitely generated submodule of $M$, then $M$ satisfies $A.C.C$ on prime submodules.*

**Corollary 2.16.** *Let $M$ be a serial module. If every prime submodule is a radical of a finitely generated submodule, then $M$ satisfies $FMP$ property.*

**Proposition 2.17.** *Let $M$ be a finitely generated module, which satisfies $FMP$ property. If $M$ is a semi-local module(that is, $Max(M)$ is a finite set), then $Spec(M)$ is a finite set.*

**Corollary 2.18.** *If $M$ is a semi-local finitely generated module which has $FMP$ property, then $Min(M)$ is a finite set. Also for each proper submodule $N$ of $M$, the number of minimal prime submodules over $N$ is finite.*

**Lemma 2.19.** *If $M$ is a semi-local finitely generated module which has $FMP$ property, then every prime submodule is a radical of a finitely generated submodule.*

The following is the the generalization of Cohen's Theorem that we have promised:

**Theorem 2.20.** *Let $M$ be a finitely generated semi-local module which satisfies $FMP$ property. If rad $N$ is finitely generated for each finitely generated submodule $N$ of $M$, then $M$ is a Noetherian module.*

## References

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