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A NOTE ON QUOTIENT HYPER K-ALGEBRAS

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Abstract. In this manuscript, we show that if $H \overset{\mathrm{H}}{=} A \oplus B$, then H/A is isomorphic to B, where H is a hyper K-algebra, A and B are closed sets.

1. Introduction

The study of BCK-algebra was initiated by Imai and Iséki in 1966. Borzooei, Zahedi and et.al. in 2000 introduced and studied the concept of hyper BCK-algebra and K-algebra.

Definition 1.1. [2] Let H be a set containing a constant "0" and "$\circ$" be a hyperoperation (i.e. a map $\circ : H \times H \to \mathcal{P}^*(H)$) on H. Then H is called a hyper K-algebra (hyper BCK-algebra) if it satisfies K1-K5(HK1-HK4).

K1: $(x \circ z) \circ (y \circ z) < x \circ y$, HK1: $(x \circ z) \circ (y \circ z) \ll x \circ y$,
K2: $(x \circ y) \circ z = (x \circ z) \circ y$, HK2: $(x \circ y) \circ z = (x \circ z) \circ y$,
K3: $x < x$, HK3: $x \circ H \ll x$,
K4: $x < y,\ y < x$ then $x = y$, HK4: $x \ll y,\ y \ll x$ then $x = y$,
K5: $0 < x$,

for all $x, y, z \in H$, where $x < y (x \ll y)$ means $0 \in x \circ y$. Moreover for any $A, B \subseteq H$, $A < B$ (resp. $A \ll B$) if there exist $a \in A$ and $b \in B$ such that $a < b$ (resp. if for all $a \in A$, there exists $b \in B$ such that $a \ll b$). If $A, B \subseteq H$, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ of H.

Definition 1.2. [2, 5] Let I be a non-empty subset of H such that $0 \in I$. Then I is said to be a *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ implies that $x \in I$ for all $x, y \in H$ and it is *closed* if for every $x, y \in H$, $x < y$ and $y \in I$ imply that $x \in A$.

Theorem 1.3. [4] *Any hyper K-ideal of hyper K-algebra H is a closed set.*

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Definition 1.4. Let I be a subset of H and $\phi \neq S \subseteq H$. Then we say that I is an *S-absorbing*, if from $x \in I$ and $y \in S$, it follows $x \circ y \subseteq I$.

Theorem 1.5. [4] *Let H be a hyper BCK-algebra and I be a hyper BCK-ideal of hyper K-algebra H. Then I is an H-absorbing.*

Definition 1.6. A hyper K-algebra H is called *(P)-decomposable* if there exists a non-trivial family $\{A_i\}_{i\in\Lambda}$ of subsets of H with the property P such that: (i) $H \neq A_i$, (ii) $H = \cup_{i\in\Lambda} A_i$, (iii) $A_i \cap A_j = \{0\}$, $i \neq j$. for all $i \in \Lambda$,
In this case, we write $H = \oplus_{i\in\Lambda} A_i(\mathrm{P})$ and say that $\{A_i\}_{i\in\Lambda}$ is a *(P)-decomposition* for H. If each A_i , $i \in \Lambda$, is an S-absorbing set we write $H \overset{\mathrm{S}}{=} \oplus_{i\in\Lambda} A_i(\mathrm{P})$.

Definition 1.7. [4] Let $\sim$ be an equivalence relation on H and $A, B \subseteq H$. Then (i) $A \sim B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \sim b$. (ii) $A \approx B$ if and only if for all $a \in A$, there exists $b \in B$ such that $a \sim b$, and for all $b \in B$ there exists $a \in A$ such that $a \sim b$. (iii) $\sim$ is called regular if $a \sim b$ implies that $a \circ c \approx b \circ c$ and $c \circ a \approx c \circ b$, for any $a, b, c \in H$. (iv) $\sim$ is called a congruence relation on H if $a \sim b$ and $x \sim y$ then $a \circ x \approx b \circ y$. (v) $\sim$ is called good, if $a \circ b \sim \{0\}$ and $b \circ a \sim \{0\}$ implies $a \sim b$ for any, $a, b, c \in H$.

Note: Henceforth, H is a hyper K-algebra and we call an equivalence, regular and good relation by " ERG relation" , the equivalence class x by C_x and $I = C_0$.

Proposition 1.8. [4] *If $\sim$ is an ERG relation on H, then $(H/\sim, *, C_0)$ is a hyper K-algebra, where $C_x * C_y = \{C_t | t \in x \circ y\}$*

Theorem 1.9. [4] *(First isomorphism theorem)Let $f : H_1 \to H_2$ be a homomorphism, i.e. f is a map such that, $f(0) = 0$ and $f(x \circ y) = f(x) \circ f(y)$, for all $x, y \in H_1$. Then $H_1/Kerf \cong Im(f)$, where $Kerf = \{x \in H_1 : f(x) = 0\}$.*

2. Main results

In this section we investigate the characteristics of an ERG relation, $\sim$, on a hyper K-algebra. We show that any such relation is as form: $x \sim y \Rightarrow x \circ y < I$ and $y \circ x < I$, where $I = C_0$. Moreover if $H \overset{\mathrm{H}}{=} \oplus_{i\in\Lambda} A_i(\text{closed})$, then this relation change into $x, y \in I$ or $x = y \notin I$ and under some conditions this relation is unique.

Theorem 2.1. (i) *Let $\sim$ be an ERG relation on H. Then $x \sim y$ if and only if $x \circ y < I$ and $y \circ x < I$.* (ii) *Let $H \overset{\mathrm{H}}{=} A \oplus B$(closed set). Then $a \circ b < A$ and $b \circ a < A$ if and only if $a, b \in A$ or $a = b \notin A$.*

Theorem 2.2. [3] *Let $H \overset{\mathrm{H}}{=} A \oplus B$(closed set), and $u \sim v$ if and only if $u, v \in A$ or $u = v \notin A$. Then* (i) *$\sim$ is an equivalence relation on H such that $C_0 = A$*

and $|H/\sim| = |B|$. (ii) $\sim$ *is a good relation.* (iii) $\sim$ *is a congruence(regular) relation if and only if* $b \circ x = b \circ y$ *where* $x, y \in A$ *and* $b \notin A$. *(iv) if* $b \circ x = b \circ y$ *where* $x, y \in A$*, and* $b \notin A$*, then* $\sim$ *is the only ERG relation on* H *such that* $C_0 = A$*, and* $H/A \cong B$.

Consider the following hyper K-algebra:

$\circ$	0	1	2	3	4	5
0	{0}	{0}	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}	{1}	{0}
2	{2}	{2}	{0,1,2}	{2}	{2}	{0}
3	{3,4}	{3,4}	{3,4}	{0,3,4}	{0,4}	{3}
4	{4}	{4}	{4}	{4}	{0,4}	{4}
5	{5}	{5}	{5}	{5}	{5}	{0,5}.

It is clear that $H = A \oplus B$(hyper K-ideal) where $A = \{0, 1, 2\}$ and $B = \{0, 3, 4, 5\}$, are H-absorbing and $b \circ x = b \circ y$ for all $b \notin A$ and $x, y \in A$. Let $\sim = \{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (2, 0), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4), (5, 5\}$. Then by Theorem 2.2, $\sim$ is an ERG relation on H. Moreover $(H/\sim, C_0, *)$, is as follows:

$*$	C_0	C_3	C_4	C_5
C_0	$\{C_0\}$	$\{C_0\}$	$\{C_0\}$	$\{C_0\}$
C_3	$\{C_3, C_4\}$	$\{C_0, C_3, C_4\}$	$\{C_0, C_4\}$	$\{C_3\}$
C_4	$\{C_4\}$	$\{C_4\}$	$\{C_0, C_4\}$	$\{C_4\}$
C_4	$\{C_5\}$	$\{C_5\}$	$\{C_5\}$	$\{C_0, C_5\}$.

If $H = A \oplus B$(hyper K-ideal) and at least one of A or B are not H-absorbing, then there is an ERG relation different from the relation given in Theorem 2.2, on H. To this end, consider the following hyper K-algebra on $H = \{0, 1, 2, 3\}$ and ERG relation $\chi = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$ 0n it. Then $H = A \oplus B$, where $A = \{0, 1\}$ and $B = \{0, 2, 3\}$. Moreover B is not H-absorbing. Also the cayley table of $H/A = \{C_0, C_2\}$ is as follows:

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2,3}	{0,2}	{1,2}
3	{3}	{3}	{0,3}	{0,2,3}.

$*$	C_0	C_2
C_0	$\{C_0\}$	$\{C_0\}$
C_2	$\{C_2\}$	$\{C_0, C_2\}$.

Now we provide an example to show that Theorem 4.1 and Corollary 4.3 of [1], are not true in general.

Remark 2.3. Theorem 4.1. and Corollary 4.3 of [1], are as follows:
"*Theorem 4.1: Let H(hyper BCK-algebra) be decomposable with decomposition* $H = A \oplus B$. *Then there exist a regular(i.e., it is a good relation according to*

the our definition) congruence relation Θ *on* H *and a hyper BCK-algebra* X *of order 2 such that* $H/\Theta \cong X$".

" *Corollary 4.3: Let* H *be decomposable with decomposition* $H = A \oplus B$ *and* $b \circ x = b \circ y$ *for all* $b \in B$ *and* $x, y \in A$*. Then* $|B| = 2$."

In the proof of Theorem 4.1 of [1], they have claimed that the relation $x\Theta y \iff x, y \in A$ *or* $x, y \in B - \{0\}$ is a congruence relation and also in Corollary 4.3 they have claimed that $|B| = 2$, but now, we show that these are not true in general. To see this, consider the following example.

Example 2.4. Consider the following hyper BCK-algebra:

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{2}
3	{3}	{3}	{3}	{0,3}

It is easy to check $H = \{0, 1\} \oplus \{0, 2, 3\}$, where $A = \{0, 1\}$ and $B = \{0, 2, 3\}$. By definition of Θ, we have $2\Theta 2$, $3\Theta 2$ but $2 \circ 3 = \{2\} \not\approx 2 \circ 2 = \{0, 2\}$, since $\{0, 2\} \not\subset A$ and $\{0, 2\} \not\subset B - \{0\}$, i.e., Θ is not a congruence relation. Also we see that $|B| \neq 2$.

Since any hyper BCK-ideal is H-absorbing, so if $H = A \oplus B$(hyper BCK-ideal), then we have $H \overset{\mathrm{H}}{=} A \oplus B$(hyper BCK-ideal). Therefore by considering Theorem 2.2, the only ERG relation on H with $C_0 = A$ is exactly as relation: $u, v \in A$ or $u = v \notin A$, hence $x\Theta y \iff x, y \in A$ *or* $x, y \in B - \{0\}$ is not a congruence relation. The correct version of Corollary 4.3. of [1] is as follows.

Theorem 2.5. *Let* H *be a BCK-algebra and* $H = A \oplus B$*(hyper BCK-ideal). Then* $H/A \cong B$.

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