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ON δ-SUPPLEMENTED MODULES

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Abstract. In this talk some characterizations of δ-supplemented and δ-lifting modules are given and are investigated some properties of these modules.

1. Introduction

Throughout this article, all rings are associative with identity, and all modules are unitary right $R-$modules. A submodule L of a module M is called *small* in M (denoted by $L \ll M$), if for every proper submodule K of M, $L + K \neq M$. A module M is called *hollow*, if every proper submodule of M is small in M.

For two submodules N and K of M, N is called a *supplement* of K in M if N is minimal with the property $M = K + N$; equivalently $M = K + N$ and $N \cap L \ll N$. A module M is called *supplemented* if every submodule of M has a *supplement* in M. Also M is called *amply supplemented* if, for any two submodules L, K of M with $M = L + K$, there exists a *supplement* P of L such that $P \leq K$. A module M is called *weakly supplemented* if, for each submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$. In this case B is called a *weak supplement* of A in M.

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *cosmall* submodule of A in M. A submodule A of M is called *coclosed* in M if A has no proper cosmall submodule. Also B is called a *coclosure* of A in M if B is a cosmall submodule of A and B is coclosed in M.

A submodule K of M is called *essential* in M (denoted by $K \leq^{ess} M$) if $K \cap X \neq 0$ for every non zero submodule X of M. We denote by $Rad(M)$ the radical of M and $R - MOD$ the category of all $R-$modules. Also we write $A \leq^{m} M$ to indicate that A is a maximal submodule of M. The *singular submodule* of a module M (denoted by $Z(M)$) is $Z(M) = \{x \in M \mid Ix = 0$ for some ideal $I \leq^{ess} R\}$. A module M is called *singular* (*nonsingular*) if $Z(M) = M$ ($Z(M) = 0$).

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Let M be a module. A submodule N of M is said to be δ-small in M(notation $N \ll_\delta M$) if, whenever $N+X=M$ with M/X singular, $X=M$. The concept of δ-small submodules was introduced by Zhou in [4]. A module M is called $\delta - hollow$, if every proper submodule of M is δ-small in M.
Every small submodule of M is δ-small in M and the converse is true whenever M is singular. But as we see in the next example the converse need not be true in general.

Example 1.1. Let R be a right semisimple ring and M be a nonzero right R-module. Then M is nonsingular and semisimple. For any nonzero $N \leq M$, N is a direct summand of M and hence is not small in M; but every submodule of M(even M itself) is δ-small in M.

Let N and L be submodules of a module M. N is called a $(weak)\delta-$*supplement* of L in M, if $N+L=M$ and $N \cap L \ll_\delta N$ ($N \cap L \ll_\delta M$). A module M is called *(weakly)δ-supplemented* if every submodule of M has a (weak)δ-supplement in M. M is called *amply δ-supplemented* if, for any submodules A and B of M with $M=A+B$, A has a δ-supplement contained in B.

2. Main results

Lemma 2.1. *Let N and L be submodules of a module M. Then the following are equivalent.*

1. *N is a δ-supplement of L in M;*
2. *$N+L=M$ and for each $K \leq N$ with $K+L=M$ and N/K singular, $K=N$.*

Lemma 2.2. *Let M be a module and $N \leq M$. Consider the following conditions:*

1. *N is a δ-supplement submodule of M;*
2. *N is weak δ-coclosed in M;*
3. *For all $x \leq M$, $x \ll_\delta M$ implies $X \ll_\delta N$.*
   Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold. If M is weakly δ-supplemented, then* (3) $\Rightarrow$ (1) *holds.*

Lemma 2.3. *For $K \subseteq L \subseteq M$, the following are equivalent:*

1. *K is a δ-cosmall submodule of L in M;*
2. *For any $X \leq M$ with M/X singular, $L+X=M$ if and only if $K+X=M$.*

Lemma 2.4. *Let M be a module. Then for any $a \in M$ we have:*
aR is not δ-small in M, if and only if there exists a maximal submodule C of M with M/C singular and $a \notin C$.

Definition 2.5. Let $\wp$ be the class of all singular simple modules. For a module M let $\delta(M) = Rej_M(\wp) = \cap\{N \subseteq M | M/N \in \wp\}$ be the reject of $\wp$ in M.

From the definition we immediately have $\delta(M/\delta(M)) = 0$, for any module M.

Proposition 2.6. *Given a module M, each of the following sets is equal to $\delta(M)$.*

(1) $A_1 = \sum\{A|A \ll_\delta M\}$.
(2) $A_2 = \cap\{B|B \leq^m M$ *with* M/B *singular*$\}$.
(3) $A_3 = \cap\{ker\phi|\phi \in Hom(M,N)$ *such that* N *is singular simple*$\}$.
(4) $A_4 = \cap\{ker\phi|\phi \in Hom(M,N)$ *such that* N *is singular semisimple*$\}$.

Proposition 2.7. *Let U and V be submodules of a module M. Assume that V is a δ-supplement of U in M. Then*

(1) *If $W + V = M$ for some $W \subseteq U$, then V is a δ-supplement of W in M,*
(2) *If $K \ll_\delta M$, then V is a δ-supplement of $U + K$ in M,*
(3) *For $K \ll_\delta M$ we have $K \cap V \ll_\delta V$ and so $\delta(V) = V \cap \delta(M)$,*
(4) *For $L \subseteq U$, $(V + L)/L$ is a δ-supplement of U/L in M/L,*
(5) *If $\delta(M) \ll_\delta M$, or $\delta(m) \subseteq U$ and if $p : M \longrightarrow M/\delta(M)$ is the canonical projection, then $M/\delta(M) = Up \oplus Vp$.*

Proposition 2.8. *Let M be an amply δ-supplemented module. Then every non δ-small submodule N of M contains a δ-supplement submodule N' such that $N/N' \ll_\delta M/N'$.*

Proposition 2.9. *For a submodule $U \subseteq M$, the following are equivalent.*

(1) *There is a direct summand X of M with $X \subseteq U$ and $U/X \ll_\delta M/X$.*
(2) *There is a direct summand $X \subseteq M$ and a submodule Y of M with $X \subseteq U$, $U = X + Y$ and $Y \ll_\delta M$.*
(3) *There is a decomposition $M = X \oplus X'$ with $X \subseteq U$ and $X' \cap U \ll_\delta X'$.*
(4) *U has a δ-supplement V in M such that $U \cap V$ is a direct summand in U.*
(5) *There is an idempotent $e \in End(M)$ with $Me \subseteq U$ and $U(1-e) \ll_\delta M(1-e)$.*

Definition 2.10. A module M is called δ-lifting if, for any $A \leq M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A/M_1 \ll_\delta M/M_1$.

For example every δ-hollow module is δ-lifting and it is easy to see that every indecomposable δ-lifting module is δ-hollow.

The next Proposition immediately follows from Proposition 2.9 and also can be found in [2, Lemma 2.3]:

Proposition 2.11. *For a module M the following are equivalent.*

(1) *M is δ-lifting.*
(2) *For every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll_\delta M$.*
(3) *Every submodule N of M can be written as $N = N_1 \oplus N_2$ with N_1 a direct summand of M and $N_2 \ll_\delta M$.*

Corollary 2.12. *Every direct summand of a δ-lifting module is δ-lifting.*

Proposition 2.13. *Let M be a δ-lifting module. Then*

(1) *Any δ-coclosed submodule of M is a direct summand;*

(2) *M is amply δ-supplemented;*
(3) *If $N \subseteq M$ is a fully invariant submodule of M, then M/N is a δ-lifting module.*

Proposition 2.14. *Let M be an amply δ-supplemented module such that every δ-supplement submodule of M is a direct summand. Then M is δ-lifting.*

Proposition 2.15. *Let M be a module such that every δ-supplement submodule of M is δ-coclosed in M. Then M is δ-lifting if and only if M is amply δ-supplemented and every δ-supplement submodule is a direct summand.*

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