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ON δ -SUPPLEMENTED MODULES

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ABSTRACT. In this talk some characterizations of δ -supplemented and δ -lifting modules are given and are investigated some properties of these modules.

1. INTRODUCTION

Throughout this article, all rings are associative with identity, and all modules are unitary right R-modules. A submodule L of a module M is called *small* in M (denoted by $L \ll M$), if for every proper submodule K of M, $L + K \neq M$. A module M is called *hollow*, if every proper submodule of M is small in M.

For two submodules N and K of M, N is called a *supplement* of K in M if N is minimal with the property M = K + N; equivalently M = K + N and $N \cap L \ll N$. A module M is called *supplemented* if every submodule of M has a *supplement* in M. Also M is called *amply supplemented* if, for any two submodules L, K of M with M = L + K, there exists a *supplement* P of L such that $P \leq K$. A module M is called *weakly supplemented* if, for each submodule A of M, there exists a submodule B of M such that M = A + B and $A \cap B \ll M$. In this case B is called a *weak supplement* of A in M.

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *cosmall* submodule of A in M. A submodule A of M is called *coclosed* in M if A has no proper cosmall submodule. Also B is called a *coclosure* of A in M if B is a cosmall submodule of A and B is coclosed in M.

A submodule K of M is called *essential* in M (denoted by $K \leq^{ess} M$) if $K \cap X \neq 0$ for every non zero submodule X of M. We denote by Rad(M) the radical of M and R - MOD the category of all R-modules. Also we write $A \leq^{m} M$ to indicate that A is a maximal submodule of M. The singular submodule of a module M (denoted by Z(M)) is $Z(M) = \{x \in M \mid Ix = 0 \text{ for some ideal } I \leq^{ess} R\}$. A module M is called singular (nonsingular) if Z(M) = M (Z(M) = 0).

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Let M be a module. A submodule N of M is said to be δ -small in M(notation $N \ll_{\delta} M$) if, whenever N + X = M with M/X singular, X = M. The concept of δ -small submodules was introduced by Zhou in [4]. A module M is called $\delta - hollow$, if every proper submodule of M is δ -small in M.

Every small submodule of M is δ -small in M and the converse is true whenever M is singular. But as we see in the next example the converse need not be true in general.

Example 1.1. Let R be a right semisimple ring and M be a nonzero right R-module. Then M is nonsingular and semisimple. For any nonzero $N \leq M$, N is a direct summand of M and hence is not small in M; but every submodule of M(even M itself) is δ -small in M.

Let N and L be submodules of a module M. N is called a $(weak)\delta$ – supplement of L in M, if N + L = M and $N \cap L \ll_{\delta} N$ $(N \cap L \ll_{\delta} M)$. A module M is called $(weakly)\delta$ -supplemented if every submodule of M has a $(weak)\delta$ -supplement in M. M is called amply δ -supplemented if, for any submodules A and B of M with M = A + B, A has a δ -supplement contained in B.

2. Main results

Lemma 2.1. Let N and L be submodules of a module M. Then the following are equivalent.

- (1) N is a δ -supplement of L in M;
- (2) N + L = M and for each $K \leq N$ with K + L = M and N/K singular, K = N.

Lemma 2.2. Let M be a module and $N \leq M$. Consider the following conditions:

- (1) N is a δ -supplement submodule of M;
- (2) N is weak δ -coclosed in M;
- (3) For all $x \leq M$, $x \ll_{\delta} M$ implies $X \ll_{\delta} N$.
- Then $(1) \Rightarrow (2) \Rightarrow (3)$ hold. If M is weakly δ -supplemented, then $(3) \Rightarrow (1)$ holds.

Lemma 2.3. For $K \subseteq L \subseteq M$, the following are equivalent:

- (1) K is a δ -cosmall submodule of L in M;
- (2) For any $X \leq M$ with M/X singular, L + X = M if and only if K + X = M.

Lemma 2.4. Let M be a module. Then for any $a \in M$ we have: aR is not δ -small in M, if and only if there exists a maximal submodule C of M with M/C singular and $a \notin C$.

Definition 2.5. Let \wp be the class of all singular simple modules. For a module M let $\delta(M) = \operatorname{Rej}_M(\wp) = \cap \{N \subseteq M | M/N \in \wp\}$ be the reject of \wp in M.

From the definition we immediately have $\delta(M/\delta(M)) = 0$, for any module M.

Proposition 2.6. Given a module M, each of the following sets is equal to $\delta(M)$.

- (1) $A_1 = \sum \{A | A \ll_{\delta} M \}.$
- (2) $A_2 = \bigcap \{B | B \leq^m M \text{ with } M/B \text{ singular} \}.$
- (3) $A_3 = \cap \{ ker\phi | \phi \in Hom(M, N) \text{ such that } N \text{ is singular simple} \}.$
- (4) $A_4 = \cap \{ ker\phi | \phi \in Hom(M, N) \text{ such that } N \text{ is singular semisimple} \}.$

Proposition 2.7. Let U and V be submodules of a module M. Assume that V is a δ -supplement of U in M. Then

- (1) If W + V = M for some $W \subseteq U$, then V is a δ -supplement of W in M,
- (2) If $K \ll_{\delta} M$, then V is a δ -supplement of U + K in M,
- (3) For $K \ll_{\delta} M$ we have $K \cap V \ll_{\delta} V$ and so $\delta(V) = V \cap \delta(M)$,
- (4) For $L \subseteq U$, (V + L)/L is a δ -supplement of U/L in M/L,
- (5) If $\delta(M) \ll_{\delta} M$, or $\delta(m) \subseteq U$ and if $p: M \longrightarrow M/\delta(M)$ is the canonical projection, then $M/\delta(M) = Up \oplus Vp$.

Proposition 2.8. Let M be an amply δ -supplemented module. Then every non δ -small submodule N of M contains a δ -supplement submodule N' such that $N/N' \ll_{\delta} M/N'$.

Proposition 2.9. For a submodule $U \subseteq M$, the following are equivalent.

- (1) There is a direct summand X of M with $X \subseteq U$ and $U/X \ll_{\delta} M/X$.
- (2) There is a direct summand $X \subseteq M$ and a submodule Y of M with $X \subseteq U, U = X + Y$ and $Y \ll_{\delta} M$.
- (3) There is a decomposition $M = X \oplus X'$ with $X \subseteq U$ and $X' \cap U \ll_{\delta} X'$.
- (4) U has a δ -supplement V in M such that $U \cap V$ is a direct summand in U.
- (5) There is an idempotent $e \in End(M)$ with $Me \subseteq U$ and $U(1-e) \ll_{\delta} M(1-e)$.

Definition 2.10. A module M is called δ -lifting if, for any $A \leq M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A/M_1 \ll_{\delta} M/M_1$.

For example every δ -hollow module is δ -lifting and it is easy to see that every indecomposable δ -lifting module is δ -hollow.

The next Proposition immediately follows from Proposition 2.9 and also can be found in [2, Lemma 2.3]:

Proposition 2.11. For a module M the following are equivalent.

(1) M is δ -lifting.

(2) For every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll_{\delta} M$.

(3) Every submodule N of M can be written as $N = N_1 \oplus N_2$ with N_1 a direct summand of M and $N_2 \ll_{\delta} M$.

Corollary 2.12. Every direct summand of a δ -lifting module is δ -lifting.

Proposition 2.13. Let M be a δ -lifting module. Then

(1) Any δ -coclosed submodule of M is a direct summand;

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- (2) M is amply δ -supplemented;
- (3) If $N \subseteq M$ is a fully invariant submodule of M, then M/N is a δ -lifting module.

Proposition 2.14. Let M be an amply δ -supplemented module such that every δ -supplement submodule of M is a direct summand. Then M is δ -lifting.

Proposition 2.15. Let M be a module such that every δ -supplement submodule of M is δ -coclosed in M. Then M is δ -lifting if and only if M is amply δ -supplemented and every δ -supplement submodule is a direct summand.

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