

Tarbiat Moallem University, 20th Seminar on Algebra,
2-3 Ordibehesht, 1388 (Apr. 22-23, 2009) pp 159-161

A GENERALIZATION OF h -DIVISIBLE MODULES

REZA NIKANDISH

Faculty of Mathematics
K. N. Toosi University of Technology
Tehran, Iran r_nikandish@sina.kntu.ac.ir
(Joint work with Mohammad. J. Nikmehr)

ABSTRACT. In this paper the notion of hw -divisible modules as a generalization of h -divisible modules is defined. We show that some of the most important properties of h -divisible modules are hold for hw -divisible modules.

1. INTRODUCTION

In this talk, R will denote a commutative domain with identity and $Q(\neq R)$ its field of quotients. The R -module $\frac{Q}{R}$ will be denoted by K . Matlis in [5] introduced the notion of h -divisible modules. Recall that an R -module is said to be h -divisible if it is a homomorphic image of an injective R -module. Lee in [4] defined the notion of weak-injective modules. An R -module M is called weak-injective if $\text{Ext}_R^1(N, M) = 0$ for all R -modules N of weak dimension ≤ 1 . He proved that the class of weak-injectives lies strictly between the classes of h -divisible and injective R -modules. For their main properties we refer to [3] and [4]. We say that an R -module D is *hw-divisible* if it is an epic image of a weak-injective R -module. Note that hw -divisible R -modules are always divisible, but not injective in general. In this paper, the definition and some general results of hw -divisible modules are given. Also, we study the relationship between hw -divisible modules and some of various generalizations of injective R -modules. Moreover, some results related to h -divisible modules are extended to hw -divisible modules. In particular, it is shown that an R -module M satisfies $\text{Ext}_R^1(M, D) = 0$ for all hw -divisible R -modules D if and only if $p.d(M) \leq 1$.

Throughout this paper the symbol of $D(R)$ stands for the global dimension. Moreover, $i.d(M)$, $p.d(M)$ and $w.dh(M)$ denote the injective, projective and weak dimension of M , respectively. The character module $\text{Hom}_{\mathbb{Z}}(M, \frac{Q}{\mathbb{Z}})$ of an R -module M will be denoted by M^b .

2000 Mathematics Subject Classification: 13C11; 13G05.

keywords and phrases: h -divisible modules, hw -divisible modules, weak injective modules.

2. MAIN RESULTS

Definition 2.1. An R -module D is called *hw-divisible* if it is an epic image of a weak-injective R -module.

Remark 2.2. It is easy to check that the implications *injective* \Rightarrow *weak-injective* \Rightarrow *h-divisible* \Rightarrow *hw-divisible* \Rightarrow *divisible* hold.

Theorem 2.3. Let R be a Matlis domain and D be an R -module. Then the following statements are equivalent.

- (a) $\text{Ext}_R^1(K, D) = 0$;
- (b) Every R -homomorphism from R in to D can be extended to an R -homomorphism from Q into D ;
- (c) D is a homomorphic image of a weak-injective R -module.

Corollary 2.4. An R -module D is hw-divisible whenever $\text{Ext}_R^1(K, D) = 0$.

Lemma 2.5. For a domain R , the following are equivalent.

- (a) R is a Dedekind domain;
- (b) All hw-divisible R -modules are injective;
- (c) All divisible R -modules are injective.

Lemma 2.6. (a) An R -module M satisfies $\text{Ext}_R^1(M, D) = 0$ for all hw-divisible R -modules D if and only if $p.d(M) \leq 1$;

(b) For any hw-divisible R -module D , $i.d(D) + 1 \leq D(R)$.

Proof. (a) From the exact sequence $0 \rightarrow H \rightarrow G \rightarrow D \rightarrow 0$ where G is a weak-injective R -module, we have $\text{Ext}_R^1(M, D) \cong \text{Ext}_R^2(M, H)$. Here the second Ext is zero whenever $p.d(M) \leq 1$. Conversely, let M have the indicated property. Given any R -module N , consider the exact sequence $0 \rightarrow N \rightarrow E \rightarrow D \rightarrow 0$ where E denotes the injective hull of N . As D is hw-divisible, we have $0 = \text{Ext}_R^1(M, D) \rightarrow \text{Ext}_R^2(M, N) \rightarrow \text{Ext}_R^2(M, E) = 0$ whence $p.d(M) \leq 1$ is evident.

(b) follows from (a) by an obvious dimension shifting argument. \square

Recall that an R -submodule N of M is called pure in M if, for all (finitely presented) R -modules F , the map $F \otimes_R N \rightarrow F \otimes_R M$ induced by the inclusion map $N \rightarrow M$ is injective. An exact sequence $0 \rightarrow N \xrightarrow{\varphi} M \rightarrow L \rightarrow 0$ is called a pure-exact sequence if $\text{Im } \varphi$ is pure in M . An R -module M is called pure-injective if it has the injective property relative to all pure-exact sequences.

Lemma 2.7. For a pure-injective R -module D , the following are equivalent.

- (a) D is divisible;
- (b) D is hw-divisible;
- (c) D is weak-injective.

Proof. We have only to prove (a) \Rightarrow (c). Let D be a divisible R -module. Then D^b is torsion-free and hence D^{bb} is weak-injective by [4, Lemma 3.1]. The pure-embedding $D \rightarrow D^{bb}$ together with the hypothesis imply that D is a summand of D^{bb} . Hence D is weak-injective. \square

Corollary 2.8. An R -module D is hw-divisible if and only if $\text{Tor}_1^R(M, D^b) = 0$, for all R -modules M of weak dimension ≤ 1 .

Proof. \Leftarrow See Lemma 2.7.

\Rightarrow Let D be an hw -divisible R -module. Then D^b is torsion-free and hence D^{bb} is weak-injective and the result can be found by the isomorphism

$$0 = \text{Ext}_R^1(N, \text{Hom}_R(D^b, \frac{\mathbb{Q}}{\mathbb{Z}})) \cong \text{Hom}_{\mathbb{Z}}(\text{Tor}_1^R(N, D^b), \frac{\mathbb{Q}}{\mathbb{Z}}).$$

□

Lemma 2.9. *For a domain R , the following are equivalent.*

- (a) *All hw -divisible R -modules are pure-injective;*
- (b) *Every hw -divisible module is a summand of A^b for some torsion-free module A .*

Recall that an R -module M is called *FP-injective* (or *absolutely pure*) if $\text{Ext}_R^1(N, M) = 0$ for all finitely presented R -modules N .

Proposition 2.10. *For a domain R , the following are equivalent.*

- (a) *R is a prüfer domain;*
- (b) *All Divisible R -modules are FP-injective;*
- (c) *All hw -divisible R -modules are FP-injective.*

Proof. (a) \Rightarrow (b) is evident as divisible submodules are relatively divisible, and hence pure in the prüfer case.

(b) \Rightarrow (a) is trivial.

(c) \Rightarrow (a) Let A be any R -module and B its injective hull. For a finitely presented R -module F , the exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ induces the exact sequence $0 = \text{Ext}_R^1(F, C) \rightarrow \text{Ext}_R^2(F, A) \rightarrow \text{Ext}_R^2(F, B) = 0$, where the first Ext vanishes by assumption. Hence $\text{Ext}_R^2(F, A) = 0$. Since A was arbitrary, we conclude $\text{pd}(F) \leq 1$. If this inequality holds for all finitely presented R -modules F , then for every finitely generated ideal I of R , $\frac{R}{I}$ has projective dimension ≤ 1 , hence I is projective; whence R must be a prüfer domain. □

Corollary 2.11. *For a domain R , the following are equivalent.*

- (a) *R is a prüfer domain;*
- (b) *All hw -divisible R -modules are FP-injective;*
- (c) *All torsion-free R -modules are flat.*

Corollary 2.12. *For a domain R , the following are equivalent:*

- (a) *Every R -module of weak dimension ≤ 1 has projective dimension ≤ 1 ;*
- (b) *All hw -divisible R -modules are weak-injective;*
- (c) *All divisible R -modules are weak-injective;*
- (d) *Epic images of weak-injective R -modules are weak-injective.*

REFERENCES

- [1] E. Fuchs and L. Scale, *A Modules over Non-Noetherian domains*, Math surveys and Monographs 84, Providence R. I.: Amer. Math. Society., 2001.
- [2] S. B. Lee *divisible modules*, Comm. Alg., **31** (2003), 513-525.
- [3] S. B. Lee *A note on the Matlis category equivalence*, J. Alg., **299** (2006), 854-862.
- [4] S. B. Lee *Weak-injective modules*, Comm. Alg., **34** (2006), 361-370.
- [5] E. Matlis *Cotorsion modules*, Mem. Amer. Math. Soc., **49**(1964).