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A GENERALIZATION OF h-DIVISIBLE MODULES

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Abstract. In this paper the notion of hw-divisible modules as a generalization of h-divisible modules is defined. We show that some of the most important properties of h-divisible modules are hold for hw-divisible modules.

1. Introduction

In this talk, R will denote a commutative domain with identity and $Q(\neq R)$ its field of quotients. The R-module $\frac{Q}{R}$ will be denoted by K. Matlis in [5] introduced the notion of h-divisible modules. Recall that an R-module is said to be h-divisible if it is a homomorphic image of an injective R-module. Lee in [4] defined the notion of weak-injective modules. An R-module M is called weak-injective if $Ext^1_R(N, M) = 0$ for all R-modules N of weak dimension ≤ 1. He proved that the class of weak-injectives lies strictly between the classes of h-divisible and injective R-modules. For their main properties we refer to [3] and [4]. We say that an R-module D is *hw-divisible* if it is an epic image of a weak-injective R-module. Note that hw-divisible R-modules are always divisible, but not injective in general. In this paper, the definition and some general results of hw-divisible modules are given. Also, we study the relationship between hw-divisible modules and some of various generalizations of injective R-modules. Moreover, some results related to h-divisible modules are extended to hw-divisible modules. In particular, it is shown that an R-module M satisfies $Ext^1_R(M, D) = 0$ for all hw-divisible R-modules D if and only if $p.d(M) \leq 1$.

Throughout this paper the symbol of $D(R)$ stands for the global dimension. Moreover, $i.d(M)$, $p.d(M)$ and $w.dh(M)$ denote the injective, projective and weak dimension of M, respectively. The character module $Hom_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}})$ of an R-module M will be denoted by M^b.

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2. Main results

Definition 2.1. *An R-module D is called hw-divisible if it is an epic image of a weak-injective R-module.*

Remark 2.2. *It is easy to check that the implications injective $\Rightarrow$ weak-injective $\Rightarrow$ h-divisible $\Rightarrow$ hw-divisible $\Rightarrow$ divisible hold.*

Theorem 2.3. *Let R be a Matlis domain and D be an R-module. Then the following statements are equivalent.*
(a) $Ext^1_R(K, D) = 0$;
(b) *Every R-homomorphism from R in to D can be extended to an R-homomorphism from Q into D;*
(c) *D is a homomorphic image of a weak-injective R-module.*

Corollary 2.4. *An R-module D is hw-divisible whenever $Ext^1_R(K, D) = 0$.*

Lemma 2.5. *For a domain R, the following are equivalent.*
(a) *R is a Dedekind domain;*
(b) *All hw-divisible R-modules are injective;*
(c) *All divisible R-modules are injective.*

Lemma 2.6. (a) *An R-module M satisfies $Ext^1_R(M, D) = 0$ for all hw-divisible R-modules D if and only if $p.d(M) \leq 1$;*
(b) *For any hw-divisible R-module D, $i.d(D) + 1 \leq D(R)$.*

Proof. (a) From the exact sequence $0 \to H \to G \to D \to 0$ where G is a weak-injective R-module, we have $Ext^1_R(M, D) \cong Ext^2_R(M, H)$. Here the second Ext is zero whenever $p.d(M) \leq 1$. Conversely, let M have the indicated property. Given any R-module N, consider the exact sequence $0 \to N \to E \to D \to 0$ where E denotes the injective hull of N. As D is hw-divisible, we have $0 = Ext^1_R(M, D) \to Ext^2_R(M, N) \to Ext^2_R(M, E) = 0$ whence $p.d(M) \leq 1$ is evident.
(b) follows from (a) by an obvious dimension shifting argument. □

Recall that an R-submodule N of M is called pure in M if, for all (finitely presented) R-modules F, the map $F \otimes_R N \to F \otimes_R M$ induced by the inclusion map $N \to M$ is injective. An exact sequence $0 \to N \xrightarrow{\varphi} M \to L \to 0$ is called a pure-exact sequence if Im φ is pure in M. An R-module M is called pure-injective if it has the injective property relative to all pure-exact sequences.

Lemma 2.7. *For a pure-injective R-module D, the following are equivalent.*
(a) *D is divisible;*
(b) *D is hw-divisible;*
(c) *D is weak-injective.*

Proof. We have only to prove (a) $\Rightarrow$ (c). Let D be a divisible R-module. Then D^b is torsion-free and hence D^{bb} is weak-injective by [4, Lemma 3.1]. The pure-imbedding $D \to D^{bb}$ together with the hypothesis imply that D is a summand of D^{bb}. Hence D is weak-injective. □

Corollary 2.8. *An R-module D is hw-divisible if and only if $Tor^R_1(M, D^b) = 0$, for all R-modules M of weak dimension ≤ 1.*

Proof. $\Leftarrow$ See Lemma 2.7.
$\Rightarrow$ Let D be an hw-divisible R-module. Then D^b is torsion-free and hence D^{bb} is weak-injective and the result can be found by the isomorphism

$$0 = Ext^1_R(N, Hom_R(D^b, \frac{\mathbb{Q}}{\mathbb{Z}})) \cong Hom_{\mathbb{Z}}(Tor^R_1(N, D^b), \frac{\mathbb{Q}}{\mathbb{Z}}).$$

□

Lemma 2.9. *For a domain R, the following are equivalent.*
(a) *All hw-divisible R-modules are pure-injective;*
(b) *Every hw-divisible module is a summand of A^b for some torsion-free module A.*

Recall that an R-module M is called *FP-injective*(or *absolutely pure*) if $Ext^1_R(N, M) = 0$ for all finitely presented R-modules N.

Proposition 2.10. *For a domain R, the following are equivalent.*
(a) *R is a prüfer domain;*
(b) *All Divisible R-modules are FP-injective;*
(c) *All hw-divisible R-modules are FP-injective.*

Proof. (a) $\Rightarrow$ (b) is evident as divisible submodules are relatively divisible, and hence pure in the prüfer case.
(b) $\Rightarrow$ (a) is trivial.
(c) $\Rightarrow$ (a) Let A be any R-module and B its injective hull. For a finitely presented R-module F, the exact sequence $0 \to A \to B \to C \to 0$ induces the exact sequence $0 = Ext^1_R(F, C) \to Ext^2_R(F, A) \to Ext^2_R(F, B) = 0$, where the first Ext vanishes by assumption. Hence $Ext^2_R(F, A) = 0$. Since A was arbitrary, we conclude $pd(F) \leq 1$. If this inequality holds for all finitely presented R-modules F, then for every finitely generated ideal I of R, $\frac{R}{I}$ has projective dimension ≤ 1, hence I is projective; whence R must be a prüfer domain. □

Corollary 2.11. *For a domain R, the following are equivalent.*
(a) *R is a prüfer domain;*
(b) *All hw-divisible R-modules are FP-injective;*
(c) *All torsion-free R-modules are flat.*

Corollary 2.12. *For a domain R, the following are equivalent:*
(a) *Every R-module of weak dimension ≤ 1 has projective dimension ≤ 1;*
(b) *All hw-divisible R-modules are weak-injective;*
(c) *All divisible R-modules are weak-injective;*
(d) *Epic images of weak-injective R-modules are weak-injective.*

References

[1] E. Fuchs and L. Scale, *A Modules over Non-Noetherian domains*, Math surveys and Monographs 84, Providence R. I.: Amer. Math. Society., 2001.
[2] S. B. Lee *divisible modules*, Comm. Alg., **31** (2003), 513-525.
[3] S. B. Lee *A note on the Matlis category equivalence*, J. Alg., **299** (2006), 854-862.
[4] S. B. Lee *Weak-injective modules*, Comm. Alg., **34** (2006), 361-370.
[5] E. Matlis *Cotorsion modules*, Mem. Amer. Math. Soc., **49**(1964).