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# PRÉSIMPLIFIABLE AND WEAKLY PRÉSIMPLIFIABLE MODULES

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ABSTRACT. In [2] it was noted that in commutative rings with zero divisors the equivalent definitions of associates in integral domains are not equivalent anymore and thus lead to different types of irreducible elements. There it was proved that in such rings these different types of associates are equivalent if and only if the ring, $R$, is présimplifiable, that is for every $r, a \in R$, $ra = a$ implies $a = 0$ or $r \in U(R)$. In [3] they extended the basic definitions of the theory of factorization to modules; including présimplifiablity. Here we define the weakly présimplifiable condition for modules. This notion enables us to reduce checking présimplifiablity and several other factorization properties in general modules to checking them just in faithful modules. Also we study how these properties behave under several module constructions.

## 1. INTRODUCTION

Throughout this paper all rings are assumed to be commutative with identity and all modules are assumed to be unitary and nonzero. D. D. Anderson and S. Valdes-Leon in [2, 3] extended the theory of factorization to commutative rings with zero divisors and modules over them. In that context equivalent definitions of irreducible elements in integral domains lead to several types of irreducible elements and hence several kinds of atomic rings (an atomic ring is one in which every element can be factorized into product of irreducible elements). But all these different types of irreducibility and atomicity turn out to be equivalent in présimplifiable rings. Here we bring some of their definitions (in the module case) and propositions. In what follows we assume that $R$ is a ring and $M$ is a left $R$-module. We denote the set of units of $R$, Jacobson radical and nilradical of $R$ by $U(R)$, $\mathrm{J}(R)$ and $N(R)$ respectively. Also $\mathrm{Z}(M)$ means the set of zero divisors of $M$, that is $\{r \in R \mid \exists 0 \neq m \in M : rm = 0\}$. Any other undefined notation is as [4].

**Definition 1.1.** Let $m, n \in M$. Then $m$ and $n$ are **associates**, denoted $m \sim n$, if $Rm = Rn$. If $m = un$ for some unit $u \in R$, then $m$ and $n$ are **strong associates**, denoted $m \approx n$. Finally we say that $m$ and $n$ are **very strong associates**, denoted $m \cong n$, if $m \sim n$ and either $m = n = 0$ or $m \neq 0$ and $m = rn$ implies $r \in U(R)$. By being associate in $R$ we mean being associate in $R$ as an $R$-module (similarly for other kinds of associativity).

A nonzero $m \in M$ is called **primitive** (resp. **strongly primitive**, **very strongly primitive**) if for $r \in R$ and $n \in M$, $m = rn$ implies $m \sim n$ (resp. $m \approx n$, $m \cong n$).

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There are two forms of factorization in modules, one in which the ring elements are not factorized and one in which they are. The second is the composition of first and factorization in rings, so we focus on the first. In the rest of this section we assume $J = \{$primitive, strongly primitive, very strongly primitive$\}$.

**Definition 1.2.** Let $\alpha \in J$ and $m \in M$. A factorization $m = an$ where $a \in R$ and $n \in M$ is called an **$\alpha$-factorization** if n is $\alpha$. We say $M$ is **$\alpha$-atomic** if every nonzero element of $M$ has an $\alpha$-factorization.

Two $\alpha$-factorizations $an_1 = an_2$ of $0 \neq m \in M$ are **isomorphic** (resp. **strongly isomorphic,very strongly isomorphic**) if $n_1 \sim n_2$ (resp. $n_1 \approx n_2$, $n_1 \cong n_2$).

Let $K = \{$isomorphic, strongly isomorphic,very strongly isomorphic$\}$.

**Definition 1.3.** Let $\alpha \in J$, $\beta \in K$. $M$ is an **$(\alpha,\beta)$-unique factorization module ($(\alpha,\beta)$-UFM)** if (1) $M$ is $\alpha$-atomic and (2) any two $\alpha$-factorization of a nonzero element of $M$ are $\beta$. And $M$ is a **bounded factorization module (BFM)** when for each $0 \neq m \in M$, there is an $N_m \in \mathbb{N}$ such that if $m = a_1 \cdots a_k n$ for nonunits $a_i \in R$ and $n \in M$, then $k \leq N_m$. $R$ is a **bounded factorization ring (BFR)** if it is a BFM as a module over itself.

**Definition 1.4.** A ring $R$ (resp. an $R$-module $M$) is called **présimplifiable** when for every $r, a \in R$ (resp. $r \in R$, $a \in M$) such that $ra = a$, we have $a = 0$ or $r \in U(R)$.

**Proposition 1.5.** *$R$ (or $M$) is présimplifiable if and only if all kinds of associates in it are equivalent and hence in a présimplifiable module all kinds of primitives, atomicities and also different types of being isomorphic, are equivalent.*

**Proposition 1.6.** *If $R$ (resp. $M$) is a BFR (resp. BFM) then it is présimplifiable.*

**Proposition 1.7.** *$M$ is présimplifiable if and only if $Z(M) \subseteq J(R)$.*

## 2. Weakly Présimplifiable Modules

**Definition 2.1.** $M$ is a **weakly présimplifiable module** when $rm = m$ for all $m \in M$ implies $r \in U(R)$.

$M$ is **r-présimplifiable** if for each $0 \neq m \in M$ and $r \in R - \mathrm{Z}(M)$, $rm = m$ implies $r \in U(R)$.

**Theorem 2.2.** *$M$ is weakly présimplifiable if and only if $Ann(M) \subseteq J(R)$.*

**Corollary 2.3.** *Every $R$-module is weakly présimplifiable iff every finitely generated $R$-module is weakly présimplifiable, iff every cyclic $R$-module is weakly présimplifiable, iff $R$ is local (note that we assumed every module is nonzero).*

**Definition 2.4.** Let $\alpha \in J$, $\beta \in K$ ($J$ & $K$ as in the introduction). We say M is a **$\beta$-weak finite factorization module ($\beta$-WFFM)** when for any $0 \neq m \in M$, up to $\beta$, there are a finite number of factorizations $m = an$ for some $a \in R$, $n \in M$. We say M is **$(\alpha,\beta)$-finite atomic factorization ($(\alpha,\beta)$-FAF)** if M is $\alpha$-atomic and the number of $\alpha$-factorizations of any nonzero element of $M$ is finite, up to $\beta$.

**Theorem 2.5.** *Let $M$ be an $R$-module, $\alpha \in J$, $\beta \in K$ ($J$ & $K$ as in the introduction).*

(1) *If $M$ is weakly présimplifiable then: $m \in M$ (taking $M$ as an $R$-module) is $\alpha$ iff it is so in $M$ as an $(R/Ann(M))$-module. $M$ as an $R$-module is $\alpha$-atomic (resp. $(\alpha,\beta)$-UFM, ($\beta$-WFFM), $(\alpha,\beta)$-FAF) iff it is so as an $(R/Ann(M))$-module.*

(2) *$M$ is présimplifiable (resp. r-présimplifiable, a BFM) iff $M$ is weakly présimplifiable and $M$ as an $(R/Ann(M))$-module is présimplifiable (resp r-présimplifiable, a BFM).*

**Corollary 2.6.** *If $M$ is présimplifiable (resp. a BFM) then $R/Ann(M)$ is présimplifiable (resp. a BFR).*

**Example 2.7.** If $R$ is a valuation ring with rank higher than one, then $R/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $R$, is a bounded factorization $R$-module although $R$ is not a BFR.

**Proposition 2.8.** *If $M$ is finitely generated then $M$ is weakly présimplifiable if and only if for any ideal $I$ in $R$, $IM = M$ implies $I = R$.*

**Corollary 2.9.** *If there exists a finitely generated weakly présimplifiable divisible $R$-module, then $R$ is a total quotient ring; in particular if $R$ is a domain over which such a module exists, it is a field.*

**Examples 2.10.** $M = \bigoplus_{n=1}^{\infty} \mathbb{Z}_{3^n}$ is a weakly présimplifiable $\mathbb{Z}$-module although $2\mathbb{Z}M = M$. $N = \mathbb{Z}_3 \bigoplus \mathbb{Z}_9$ is a finitely generated weakly présimplifiable $\mathbb{Z}_9 \bigoplus \mathbb{Z}_9$-module and there are ideals $I \neq J$ of $\mathbb{Z}_9 \bigoplus \mathbb{Z}_9$ such that $IN = JN$.

## 3. Polynomial Modules and Direct Sums

In [1] it was proved that $R[x]$ is présimplifiable iff in $R$ the zero ideal is primary. To obtain a similar result about $M[x]$ as an $R[x]$-module we need:

**Definition 3.1.** *We say $M$ is with nil zero divisors (with N.Z., for short) if $Z(M) \subseteq N(R)$. And it is with nil annihilator (abbreviated N.A.) if $Ann(M) \subseteq N(R)$.*

**Lemma 3.2.** *If $f \in Z(M[x])$ then there is an $0 \neq m \in M$ such that $fm = 0$.*

**Theorem 3.3.** *$M[x]$ as an $R[x]$-module is présimplifiable iff it is r-présimplifiable, iff it is with N.Z., iff $M$ is with N.Z..*

*$M[x]$ as an $R[x]$-module is weakly présimplifiable iff it is with N.A., iff $M$ is with N.A..*

**Proposition 3.4.** *A faithful module is with N.Z. iff its zero submodule is primary.*

$M[x]$ as an $R$-module is isomorphic to $M^{\mathbb{N}}$, so we state more generally:

**Proposition 3.5.** *Assume each $M_i$ is an $R$-module. $\bigoplus_{i\in A} M_i$ is présimplifiable (resp. with N.Z.) iff $\prod_{i\in A} M_i$ is so iff each $M_i$ is so. If any $M_i$ is weakly présimplifiable (resp. with N.A.) then $\bigoplus_{i\in A} M_i$ and $\prod_{i\in A} M_i$ are so.*

**Example 3.6.** If $R_1 = \mathbb{Z}_2$ and $R_2 = \mathbb{Z}_3$ and $R = R_1 \bigoplus R_2$ then although $R$ as an $R$-module is with N.A. but none of its submodules is even weakly présimplifiable.

**Proposition 3.7.** *Assume $R$ is a perfect ring (which means it satisfies the minimal condition on principal ideals), and $A$ is an infinite set and $M_i$ are $R$-modules. $\bigoplus_{i\in A} M_i$ (or equivalently $\prod_{i\in A} M_i$) is weakly présimplifiable if and only if there is a $F \subseteq A$ such that $|F| < \infty$ and $\bigoplus_{i\in F} M_i$ is weakly présimplifiable.*

**Proposition 3.8.** *Assume each $M_i$ is an $R_i$ module ($i \in A$). $M = \bigoplus_{i\in A} M_i$ (or equivalently $N = \prod_{i\in A} M_i$) as a $\prod_{i\in A} R_i$-module is présimplifiable (resp. with N.Z.) iff $|A| = 1$ and the only $M_i$ is présimplifiable (resp. with N.Z.). $M$ (or $N$) is weakly présimplifiable iff each $M_i$ is weakly présimplifiable. If $M$ or $N$ is with N.A. then each $M_i$ is with N.A..*

**Corollary 3.9.** *A présimplifiable ring is indecomposable.*

**Example 3.10.** If, in the notation of 3.8, $R_i = \mathbb{Z}_{2^i}$ and $M_i = R_i/N(R_i)$, then each $M_i$ is with N.A. but $M$ and $N$ are not so.

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