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RESULTS ON FUZZY FILTERS IN BL-ALGEBRAS

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Abstract. In this paper we defined the notions of fuzzy implicative filter, fuzzy positive implicative filter and fuzzy fantastic filter and we find some relations among them. Specially, we find the relations between the fuzzy implicative filters and Gödel algebras.

1. Introduction and Preliminary

Definition 1.1. [1] A BL-algebra is a structure $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ such that,
(i) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(ii) $(L, \odot, 1)$ is an Abelian monoid,
(iii) The following conditions hold for all $x, y \in L$:
(B1) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$,
(B2) $x \wedge y = x \odot (x \rightarrow y)$,
(B3) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Lemma 1.2. [1] *Let L be a BL-algebra. The following properties hold:*
(i) $x \leq y$ *if and only if* $x \rightarrow y = 1$,
(ii) $x \rightarrow (y \rightarrow z) = x \odot y \rightarrow z$,
(iii) $x \odot y \leq x \wedge y$,
(iv) $(x \rightarrow y) \odot (y \rightarrow z) \leq (x \rightarrow z)$,
(v) $x \rightarrow \neg x = \neg\neg x \rightarrow \neg x$,
(vi) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(vii) $x \vee \neg x = 1 \Rightarrow x \wedge \neg x = 0$,
(viii) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$ *where* $\neg x = x \rightarrow 0$.

Definition 1.3. $[1, 2, 3, 5, 6]$ Let L be a BL-algebra and F be a nonempty subset of L.
(i) F is called a filter of L if $1 \in F$ and if $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$,
(ii) F is called an implicative filter if $1 \in F$ and if $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ then $x \rightarrow y \in F$,
(iii) F is called a positive implicative filter if $1 \in F$ and if $x \in F$, $x \rightarrow ((y \rightarrow$

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$z) \to z) \in F$ then $y \in F$,
(iiv) Let F be a filter of L. Then F is a Boolean filter if $x \vee \neg x \in F$,
(v) Let f be a fuzzy set on L. Then f is called a fuzzy filter if $f(x) \leq f(1)$ and $f(x) \wedge f(x \to y) \leq f(y)$,
(vi) Let f be a fuzzy filter on L. Then f is called a fuzzy boolean filter if $f(x \vee \neg x) = f(1)$,
for all $x, y, z \in L$.

2. Fuzzy Implicative Filter on BL-algebras

Definition 2.1. Let f be a fuzzy filter on L. Then f is called a fuzzy implicative filter if it satisfies:
(FF1) $f(x) \leq f(1)$, for all $x \in L$,
(FF3) $f(x \to (y \to z)) \wedge f(x \to y) \leq f(x \to z)$,
for all $x, y, z \in L$.

Theorem 2.2. *Any fuzzy implicative filter is a fuzzy filter, but the converse is not correct in general.*

Theorem 2.3. *Let f be a fuzzy set on L. Then the following are equivalent:*
(i) *f is a fuzzy implicative filter,*
(ii) *f is a fuzzy filter and $f(x \to (x \to y)) \leq f(x \to y)$,*
(iii) *f is a fuzzy filter and $f(x \to (y \to z)) \leq f((x \to y) \to (x \to z))$,*
(iv) *$(FF1)$ holds and $f(z \to (y \to (y \to x))) \wedge f(z) \leq f(y \to x)$,*
(v) *$f(x \to x^2) = f(1)$ and f is a fuzzy filter.*

Theorem 2.4. *Let L be a BL-algebra and $\mu : L \longrightarrow [0,1]$ is defined by*

$$\mu(x) = \begin{cases} 0 & x \neq 1 \\ 1 & \text{x} = 1 \end{cases}$$

Then the following are equivalent:
(i) *L is a Gödel algebra,*
(ii) *Any fuzzy filter is a fuzzy implicative filter,*
(iii) *μ is a fuzzy implicative filter.*

3. Fuzzy Positive Implicative Filters on BL-algebras

Definition 3.1. Let f be a fuzzy set on L. Then f is called a fuzzy positive implicative filter if it satisfies (FF1) and
(FF4) $f(x \to ((y \to z) \to y)) \wedge f(x) \leq f(y)$, for all $x, y, z \in L$.

Theorem 3.2. *Any fuzzy positive implicative filter is a fuzzy filter.*

Theorem 3.3. *Let f be a fuzzy filter on L. Then the following condition are equivalent:*
(i) *f is a fuzzy positive implicative filter,*
(ii) *$f((x \to y) \to x) \leq f(x)$,*
for all $x, y \in L$.

Theorem 3.4. *Any fuzzy positive implicative filter is a fuzzy implicative filter.*

Theorem 3.5. *Let f be a fuzzy Implicative filter. Then f is a fuzzy positive implicative filter if and only if $f((x \to y) \to y) \leq f((y \to x) \to x)$, for all $x, y \in L$.*

Proposition 3.6. *Let f be a fuzzy filter on L. Then f is a fuzzy positive implicative filter if and only if $f((\neg x \to x) \to x) = f(1)$, for all $x \in L$.*

Theorem 3.7. *Let f be a fuzzy set. Then f is a fuzzy Boolean filter if and only if f is a fuzzy positive implicative filter.*

4. Fuzzy Fantastic Filters on BL-algebras

Definition 4.1. Let f be a fuzzy set on L. Then f is a fuzzy fantastic filter if
(FF1) $f(x) \leq f(1)$, for all $x \in L$,
(FF5) $f(z \to (y \to x)) \wedge f(z) \leq f(((x \to y) \to y) \to x)$, for all $x, y \in L$.

Proposition 4.2. *Any fuzzy fantastic filter is a fuzzy filter, but the converse is not correct in general.*

Proposition 4.3. *Let f be a fuzzy filter. Then f is a fuzzy fantastic filter if and only if*

$$f(y \to x) \leq f(((x \to y) \to y) \to x)$$

for all $x, y \in L$.

Theorem 4.4. *Any fuzzy positive implicative filter is a fuzzy fantastic filter, but the converse is not correct in general.*

Example 4.5. Let $L = \{0, a, b, 1\}$ be a chain with the following tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Let $\wedge$ and $\vee$ are defined on L as min and max, respectively. Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. Now, let the fuzzy set f on L is defined by

$$f(1) = t_2, f(b) = f(a) = f(0) = t_1 \ (0 \leq t_1 \leq t_2 \leq 1)$$

It is easily to check that f is a fuzzy fantastic filter. But f is not a positive implicative filter. Because,

$$f((b \to 0) \to b) \nleq f(b)$$

Theorem 4.6. *Let f be a fuzzy filter. Then the following conditions are equivalent:*
(i) *f is a fuzzy fantastic filter,*
(ii) *$f(((x \to 0) \to 0) \to x) = f(1)$, for all $x \in L$,*
(iii) *$f(x \to u) \wedge f(y \to u) \leq f(((x \to y) \to y) \to u)$, for all $x, y, u \in L$.*

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