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TRIANGULAR MATRIX REPRESENTATIONS OF RING EXTENSIONS

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Abstract. We investigate the class of piecewise prime, PWP, rings which properly includes all piecewise domain. For a PWP ring we determine a large class of ring extensions which have a generalized triangular matrix representation for which the diagonal rings are prime. We investigate the quasi-Baer and related conditions on monoid rings $R[G]$. In particular we consider the transfer of the quasi-Baer and related conditions between R and a u.p.-monoid ring $R[G]$ of a u.p.-monoid G over R. We characterize the semi-central idempotents of various ring extensions of R in terms of the semi-central idempotents of R.

1. Introduction

All rings are associative and R denotes a ring with unity 1. By a ring extension we mean an covering of R which has the same unity as R. An idempotent $e \in R$ is left(right) *semi-central* in R if $exe = xe$ $(exe = ex)$ for all $e \in R$. we use $S_l(R)$ and $S_r(R)$ for the sets of all left and right semi-central idempotents respectively. An idempotent e of R is *semi-central reduced* if $S_l(eRe) = \{0, e\}$. Note that $S_l(eRe) = \{0, R\}$ if and only if $S_r(eRe) = \{0, R\}$. A ring R is semi-central reduced if 1 is semi-central reduced. Also note that $S_r(R) \bigcap S_l(R) = B(R)$, where $B(R)$ is the set of all central idempotents R. A ring R has a *generalized triangular matrix representation* if there exists a ring isomorphism

$$\Theta : R \longrightarrow \begin{bmatrix} R_1 & R_{12} & . & . & . & R_{1n} \\ 0 & R_2 & . & . & . & R_{2n} \\ . & 0 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & R_n \end{bmatrix}$$ where each diagonal ring, R_i, is a

ring with unity, R_{ij} is a left R_i-right R_j-bimodule for $i < j$, and the matrices obey the usual rules for matrix addition and multiplication. If each R_i is semi-central reduced then R has a *complete generalized triangular matrix representation* with triangulating dimension n. R is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal is

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generated by an idempotent. An ordered set $\{b_1, ..., b_n\}$ of nonzero distinct idempotents in R is called a set of *left triangulating idempotents* of R if all the following hold: (i) $1 = b_1 + ... + b_n$; (ii) $b_1 \in S_l(R)$; and (iii) $b_{k+1} \in S_l(c_k R c_k)$ where $c_k = 1 - (b_1 + ... + b_k)$ for $1 \leq k \leq n-1$. A set $\{b_1, ..., b_n\}$ of left triangulating idempotents is said to be *complete* if each b_i is semi-central reduced. Any complete set of primitive idempotents determines a complete set of left triangulating idempotents. R has a (complete) set of left triangulating idempotents if and only if R has a (complete) generalized triangular matrix representation. R has *triangulating dimension* n, written $Tdim(R) = n$ if R has a complete set of left triangulating idempotents with exactly n elements. R is semi-central reduced if and only if $Tdim(R) = 1$. Let B be a set of left triangulating idempotents of R and Γ a ring extension od R. (i) We say Γ is *B-triangularly linked* to R if whenever $b \in B$ and $0 \neq c \in S_l(b\Gamma b)$, then there exists $0 \neq c_0 \in S_l(bRb)$ such that $c_0\Gamma \subseteq c\Gamma$. (ii) we say Γ is *B-triangularly compatible* with R if B is a set of left triangulating idempotents of Γ. If Γ is B-triangularly linked to (resp. triangularly compatible with) R for every set B of left triangulating idempotents , then we say Γ is *triangularly linked to*(resp. *triangularly compatible* with) R. we call $b \in R$ a *triangulating idempotent* if b is in some set of left triangulating idempotents of R. For any subset X of a ring R, $r_R(X)$ (resp. $\ell_R(X)$) denotes the right (resp. left) annihilator of X in R.

2. Main results

Theorem 2.1. *Let $R[G]$ be the monoid ring of a u.p.-monoid G over a ring R. then we have the following:* (i) *R is right p.q.-Baer if and only if $R[G]$ is right p.q.-Baer.* (ii) *R is quasi-Baer if and only if $R[G]$ is quasi-Baer.*

Theorem 2.2. *let σ be a ring endomorphism of R and X a nonempty set of not necessarily commuting indeterminates.* (i) *Let $\Gamma = R[x;\sigma]$ or $R[[x;\sigma]]$. Then $e \in S_l(\Gamma)$ with $e_0 \in R$ the constant term of e if and only if $e_0 \in S_l(R)$, $ee_0 = e_0$, $e_0e = e$.* (ii) *Let $\Delta = R[X]$ or $R[[X]]$. Then $e \in S_l(\Delta)$ with $e_0 \in R$ the constant term of e if and only if $e_0 \in S_l(R)$, $ee_0 = e_0$, $e_0e = e$.*

Theorem 2.3. *Let R be a right p.q.-Baer ring. Then $Tdim(R) = n$ if and only if R has exactly n minimal prime ideals.*

Theorem 2.4. *Let 1_M denote the unity of $Mat_n(R)$ and $\overline{R} = R.1_M$. Then $e = (e_{ij}) \in S_l(Mat_n(R))$ if and only if $e_0 \in S_l(\overline{R})$ and $e_0Mat_n(R) = eMat_n(R)$, where $e_0 = e_{11}.1_M$.*

Theorem 2.5. *Let X be a nonempty set of not necessarily commuting indeterminates, σ an endomorphism of R, and G a monoid. The following extensions of R are triangularly linked to R and triangularly compatible with R, hence have the same triangulating dimension as R:* (i) *$R[G]$ where G is a u.p.-monoid and R is a right p.q.-Baer ring;* (ii) *$R[G]$ where G is a free monoid;* (iii) *$R[X]$;* (iv) *$R[[X]]$;* (V) *$R[x, x^{-1}]$;* (vi) *$R[[x, x^{-1}]]$;* (vii)*$R[x;\sigma]$ where σ is a ring endomorphism of R such that $\sigma(bR) \subseteq bR$ for every left triangulating idempotent $b \in R$;* (viii) *$R[[x;\sigma]]$ where σ is a ring endomorphism of R such that $\sigma(bR) \subseteq bR$ for every left triangulating idempotent $b \in R$;* (iX) *$Mat_n(R)$.*

Theorem 2.6. *Let R be a quasi-Baer ring with a complete set of left triangulating idempotents $B = \{b_1, ..., b_n\}$. If Γ is any of the following ring extensions of R then Γ is a quasi-Baer ring with B determining a complete generalized triangular matrix representation for Γ in which each diagonal ring R_i is a prime ring:* (i) *$R[G]$ where G is a u.p.-monoid;* (ii) *$R[X]$ where X is a nonempty set of not necessarily commuting indeterminates;* (iii) *$R[[X]]$ where X is a nonempty set of not necessarily commuting indeterminates;* (iv) *$R[x, x^{-1}]$;* (v) *$R[[x, x^{-1}]]$;* (vi) *$R[x; \alpha]$ where α is a ring automorphism such that $\alpha(bR) \subseteq bR$ for all $b \in R$;* (vii) *$R[[x; \alpha]]$ where α is a ring automorphism such that $\alpha(bR) \subseteq bR$ for all $b \in R$;* (viii) *$T_n(R)$;* (ix)*$Mat_n(R)$.*

Theorem 2.7. *For a ring R with a complete set of triangulating idempotents, the following conditions are equivalent:* (i) *R is right p.q.-Baer;* (ii) *R is left p.q.-Baer;* (iii) *R is quasi-Baer;* (iv) *R is a pwp ring.*

Lemma 2.8. *Let $R = F[G]$ be a semi-prime group algebra over a field F. Then R is quasi-Baer if and only if each annihilator ideal is finitely generated.*

Lemma 2.9. *Let $\{b_1, ..., b_n\}$ be a set of left triangulating idempotents of R. Then $Tdim(R) = \sum_{i=1}^{n} Tdim(b_i R b_i)$. In particular $n \leq Tdim(R)$.*

Lemma 2.10. *If R satisfies any of the following conditions then R has a complete triangular matrix representations: (i) R has a complete set of primitive idempotents; (ii) R has no infinite set of orthogonal idempotents; (iii) R has D.C.C on(idempotent generated, principal, or finitely generated) ideals; (iv) R has D.C.C on(idempotent generated, principal, or finitely generated) ideals; (v) R has A.C.C on(idempotent generated, principal, or finitely generated)ideals; (vi) R has A.C.C on(idempotent generated, principal, or finitely generated) ideals; (vii) R has either A.C.C or D.C.C on right annihilators; (viii) R is a semi-primary ring.*

Lemma 2.11. *The following conditions are equivalent:* (i) *R has a complete set of left triangulating idempotents;* (ii) *$\{bR \mid b \in S_l(R)\}$ is a finite set;* (iii) *$\{bR \mid b \in S_l(R)\}$ satisfies A.C.C and D.C.C;* (iv) *$\{bR \mid b \in S_l(R)\}$ and $\{Rc \mid c \in S_r(R)\}$ satisfies A.C.C;* (v) *$\{bR \mid b \in S_l(R)\}$ and $\{Rc \mid c \in S_r(R)\}$ satisfies D.C.C;* (vi) *$\{bR \mid b \in S_l(R)\}$ and $\{cR \mid c \in S_r(R)\}$ satisfies D.C.C;* (vii) *$\{bR \mid b \in S_l(R)\}$ is a finite set;* (viii) R has a complete set of right triangulating idempotents; (ix) *R has a complete generalized triangular matrix representation;*

Lemma 2.12. *If $\{e_1, ..., e_n\}$ is a complete set of primitive idempotents R,then $Tdim(R) \leq n$.*

Lemma 2.13. *A ring R is prime if and only if is quasi-Baer and semi-central reduced.*

Lemma 2.14. *A ring R is right p.q.-Baer if and only if whenever I is a principal ideal of R there exists $e \in S_l(R)$ such that $I \subseteq Re$ and $r(I) \cap Re = (1 - e)Re$.*

Lemma 2.15. *The following conditions are equivalent:* (i) *R is a prime ring;* (ii) *R is a semi-central reduced quasi-Baer ring;* (iii) *R is a semi-central reduced p.q.-Baer ring;* (iv) *R is a semi-central reduced right p.q.-Baer ring.*

Corollary 2.16. *Any PWD (piecewise domain) is a quasi-Baer ring.*

Corollary 2.17. *If ring R is noetherian or artinian ring then R has a triangular matrix representation.*

Corollary 2.18. *Let Γ be a ring extension of R such that $\Gamma I \subseteq I\Gamma$ for all ideal I of R and whenever $b \in S_l(\Gamma)$, then there exists $b_0 \in R$ such that $b_0 b = b$ and $bb_0 = b_0$. If Γ is prime(resp. left p.q.-Baer) then R is prime(resp.left p.q.-Baer).*

Corollary 2.19. *If $Tdim(R) \lneq \infty$, then if R is prime then R is right p.q.-Baer.*

Corollary 2.20. *A ring R is prime with $Tdim(R) = m$ if and only if $T_n(R)$ is prime with $Tdim(T_n(R)) = mn$.*

Corollary 2.21. *The PWP property is a Morita invariant.*

Corollary 2.22. *Let A and B be semi-central reduced rings. If $T_m(A) \cong T_n(B)$ for some m and n, then $m = n$ and $A \cong B$*

Corollary 2.23. *The right p.q.-Baer condition is a Morita invariant property.*

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