Tarbiat Moallem University, 20^{th} Seminar on Algebra,
2-3 Ordibehesht, 1388 (Apr. 22-23, 2009) pp 200-203

CATEGORICAL PROPERTIES OF SEQUENTIALLY DENSE MONOMORPHISMS OF SEMIGROUP ACTS

LEILA SHAHBAZ

Department of Mathematical Sciences
University of Maraghe
Moalleme Shomali Ave. Maraghe, Iran leilashahbaz@yahoo.com
(Joint work with Mojgan Mahmoudi)

ABSTRACT. Let $\mathcal{M}$ be a class of (mono)morphisms in a category $\mathcal{A}$. To study mathematical notions, such as injectivity, tensor products, flatness, one needs to have some categorical and algebraic information about the pair ($\mathcal{A}$,$\mathcal{M}$).

In this paper we take $\mathcal{A}$ to be the category **Act-S** of S-acts, for a semigroup S, and $\mathcal{M}_d$ to be the class of sequentially dense monomorphisms (of interests to computer scientists, too) and study the categorical properties, such as limits and colimits, of the pair ($\mathcal{A}$,$\mathcal{M}$). Injectivity with respect to this class of monomorphisms have been studied by Giuli, Ebrahimi, and the authors and got information about injectivity relative to monomorphisms.

1. INTRODUCTION

Let $\mathcal{M}$ be a class of morphisms of a category $\mathcal{A}$. To study mathematical notions, such as injectivity and flatness, one needs to have some categorical and algebraic information about the pair ($\mathcal{A}$,$\mathcal{M}$).

In this paper we take $\mathcal{A}$ to be the category **Act-S** of (right) acts over a semigroup S and $\mathcal{M}_d$ to be the class of sequentially dense monomorphisms, to be defined in section 2, and study the categorical properties of this pair which are usually related to the behaviour of $\mathcal{M}_d$-injectivity.

In the following we first recall some facts about the category **Act-S** needed in this paper.

Let S be a semigroup and A be a set. If we have a mapping (called the *action* of S on A)

$$\begin{aligned} \mu: \quad & A \times S \to A \\ & (a,s) \longmapsto as := \mu(a,s) \end{aligned}$$

such that $a(st) = (as)t$ for $a \in A, s,t \in S$, we call A a (*right*) S-*act* or a (*right*) *act* over S.

If S is a monoid with its identity 1, we usually also require that $a1 = a$ for $a \in A$.

2000 Mathematics Subject Classification: 08B25, 18A20, 18A30, 20M30, 20M50.
keywords and phrases: Sequential closure, sequential dense.

A *homomorphism* (or an *equivariant* map, or an *S-map*) from an S-act A to an S-act B is a function from A to B such that for each $a \in A, s \in S$, $f(as) = f(a)s$.

Since the identity maps and the composition of two equivariant maps are equivariant, we have the category **Act-S** of all right S-acts and S-maps between them.

2. Sequential closure operators

In this section we introduce a closure operator, where the dense monomorphisms resulting from it is the subject of study in this paper. First note that, denoting the lattice of all subacts of an S-act B by $SubB$, we get:

Definition 2.1. The *sequential closure operator* $C^d = (C^d_B)_{B\in \mathbf{Act-S}}$ on **Act-S** is defined as

$$C^d_B(A) = \{b \in B : bS \subseteq A\}$$

for any subact A of an S-act B.

Now, one has the usual two classes of monomorphisms related to the notion of a closure operator as follows:

Definition 2.2. Let $A \leq B$ be in **Act-S**. We say that A is C^d-*closed* in B if $C^d_B(A) = A$, and it is C^d-*dense* (or *sequentially dense* or *s-dense*) in B if $C^d_B(A) = B$. Also, an S-map $f : A \to B$ is said to be C^d-dense (C^d-closed) if $f(A)$ is a C^d-dense (C^d-closed) subact of B.

We take $\mathcal{M}_d$ to be the set of all C^d-dense monomorphisms.

As the following result shows, C^d is not idempotent in general.

Theorem 2.3. *C^d is idempotent ($C^d_B(C^d_B(A)) = C^d_B(A)$ for all S-acts B and $A \leq B$) if and only if $S^2 = S$.*

3. Categorical properties of s-dense monomorphisms

In this section we study some categorical and algebraic properties of the category **Act-S** with respect to sequentially dense monomorphisms. We study the composition, limit, and colimit properties in the following three subsections.

3.1. Composition properties of s-dense monomorphisms.

The class $\mathcal{M}_d$ is clearly isomorphism closed; that is, contains all isomorphisms and is closed under composition with isomorphisms. But, unfortunately $\mathcal{M}_d$ is not always closed under composition:

Lemma 3.1. *The class $\mathcal{M}_d$ is closed under composition if and only if the C^d-closure operator is idempotent.*

Proposition 3.2. *The composition of an s-dense monomorphism with a surjective morphism is an s-dense morphism.*

Proposition 3.3. *Let $f : A \to B \in$* **Act-S**. *Then there are unique (always up to isomorphisms) morphisms $e, m \in$* **Act-S** *such that:*

(1) *(right $\mathcal{M}_d$-factorization) $f = me$ with $m \in \mathcal{M}_d$, and*

(2) *(diagonalization property of the factorization) for any commutative diagram*

$$\begin{array}{cccccc} & A & \xrightarrow{u} & D & \\ e & \downarrow & & & \\ & C & & \downarrow & g \\ m & \downarrow & & & \\ & B & \xrightarrow{v} & E & \end{array}$$

in **Act-S** *with $g : D \to E \in \mathcal{M}_d$, there is a uniquely determined morphism $w : C \to D$ with $gw = vm$ and $we = u$.*

3.2. **Limits of s-dense monomorphisms.**

Proposition 3.4. *$\mathcal{M}_d$ is closed under products.*

Proposition 3.5. *The class of sequentially dense monomorphisms is closed under $\mathcal{M}_d$-pullbacks.*

Proposition 3.6. *The class of sequentially dense monomorphisms is stable under $\mathcal{M}_d$-pullbacks; in the sense that pullback of any s-dense monomorphism along any morphism is again s-dense.*

Proposition 3.7. *$\mathcal{M}_d$ is closed under limits.*

3.3. **Colimits of s-dense monomorphisms.**

Proposition 3.8. *$\mathcal{M}_d$ is closed under coproducts.*

Proposition 3.9. *$\mathcal{M}_d$ is closed under direct sums.*

Proposition 3.10. *In* **Act-S**, *pushouts transfer s-dense monomorphisms.*

Proposition 3.11. *The pushout of s-dense monomorphisms belongs to $\mathcal{M}_d$.*

Proposition 3.12. *Multiple pushout of s-dense monomorphisms is an s-dense monomorphism. Also, multiple pushouts transfer s-dense monomorphisms.*

Definition 3.13. We say that a category $\mathcal{A}$ has $\mathcal{M}$-*bounds* if for every set indexed family $\{m_i : A \to A_i : i \in I\}$ of $\mathcal{M}$-morphisms there is an $\mathcal{M}$-morphism $m : A \to B$ which factors over all m_i's; that is there are $d_i : A_i \to B$ with $d_i m_i = m$.

Proposition 3.14. **Act-S** *has $\mathcal{M}_d$-bounds.*

Definition 3.15. We say that a category $\mathcal{A}$ has $\mathcal{M}$-*amalgamation* property, if the morphism m in the definition of $\mathcal{M}$-bounds factors over all m_i's through members of $\mathcal{M}$; that is d_i's belong to $\mathcal{M}$.

Proposition 3.16. **Act-S** *has $\mathcal{M}_d$-amalgamation property.*

Proposition 3.17. **Act-S** *has $\mathcal{M}_d$-directed colimits.*

Definition 3.18. We say that a category $\mathcal{A}$ fulfills the $\mathcal{M}$-*chain* condition if for every directed system $((A_\alpha)_{\alpha\in I}, (f_{\alpha\beta})_{\alpha\leq\beta\in I})$ whose index set I is a well-ordered chain with the least element 0, and $f_{0\alpha} \in \mathcal{M}$ for all α, there is a (so called "upper bound") family $(g_\alpha : A_\alpha \to A)_{\alpha\in I}$ with $h_0 \in \mathcal{M}$ and $g_\beta f_{\alpha\beta} = g_\alpha$.

Proposition 3.19. **Act-S** *fulfills the* $\mathcal{M}_d$*-chain condition.*

Theorem 3.20. *The class of sequentially dense monomorphisms is closed under colimits.*

References

[1] B. Banaschewski, *Injectivity and essential extensions in equational classes of algebras.* Queen's Papers in Pure and Applied Mathematics, **25** (1970), 131-147.

[2] M. Mahmoudi and L. Shahbaz, *Characterizing semigroups by sequentially dense injective acts*, Semigroup Forum **75** (2007),116-128.

[3] M. Mahmoudi and L. Shahbaz, *Well-behaviour of sequential injectivity of acts over semigroups*, to appear in Communications in Algebra.

[4] M. Mahmoudi and L. Shahbaz, *Sequential dense essential monomorphisms of acts over semigroups*, published online in Applied Categorical Structures.