Tarbiat Moallem University, 20^{th} Seminar on Algebra,
2-3 Ordibehesht, 1388 (Apr. 22-23, 2009) pp 207-209

ON SOME AUTOMORPHISMS OF A FINITE p-GROUP CENTRALIZING THE FRATTINI QUOTIENT

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ABSTRACT. Let G be a group and let $\mathrm{Aut}^{\Phi}(G)$ denote the group of all automorphisms of G centralizing $G/\Phi(G)$ elementwise. In this paper, we first characterize the finite p-groups G with cyclic Frattini subgroup for which $|\mathrm{Aut}^{\Phi}(G):\mathrm{Inn}(G)| = p$. We then give a necessary and sufficient condition on a finite p-group G for the group $\mathrm{Aut}^{\Phi}(G)$ to be elementary abelian.

1. INTRODUCTION

Let M and N be normal subgroups of a group G. We let $\mathrm{Aut}^{N}(G)$ denote the group of all automorphisms of G normalizing N and centralizing G/N, and $\mathrm{Aut}_{M}(G)$ the group of all automorphisms of G centralizing M. Moreover,

$$\mathrm{Aut}_{M}^{N}(G) = \mathrm{Aut}^{N}(G) \bigcap \mathrm{Aut}_{M}(G).$$

Müller in [3] proved, using techniques from cohomology, that if G is a finite non-abelian p-group, then $\mathrm{Aut}_{Z}^{\Phi}(G) = \mathrm{Inn}(G)$ if and only if $\Phi \leq Z$ and Φ is cyclic, where $Z = Z(G)$. This turns out that $\mathrm{Aut}^{\Phi}(G)/\mathrm{Inn}(G)$ is non-trivial if and only if G is neither elementary abelian nor extraspecial. In this paper we characterize the finite p-groups G with cyclic Frattini subgroup for which $|\mathrm{Aut}^{\Phi}(G):\mathrm{Inn}(G)| = p$.

In [2], Jafari gives a necessary and sufficient condition on a finite purely non-abelian p-group G for the group $\mathrm{Aut}^{Z}(G)$, the central automorphism of G, to be elementary abelian. We give a similar result for the $\mathrm{Aut}^{\Phi}(G)$.

2. MAIN RESULTS

Theorem 2.1. *Let G be a finite group with $\Phi(G) \leq Z(G)$. Then there is a bijection from $Hom(G/G', \Phi(G))$ onto $\mathrm{Aut}^{\Phi}(G)$ associating to every homomorphism $f: G \to \Phi(G)$ the automorphism $x \mapsto xf(x)$ of G. In particular, if G is a p-group and $\exp(\Phi(G)) = p$ then $\mathrm{Aut}^{\Phi}(G) \cong Hom(G/G', \Phi(G))$.*

Lemma 2.2. *Let G be a minimal non-abelian p-group with cyclic Frattini subgroup. Then $G \cong Q_8$ or G is one of the following groups:*

(i) $\langle x, y | x^{p^m} = y^p = 1, x^y = x^{1+p^{m-1}} \rangle$, *where* $m > 1$.

2000 Mathematics Subject Classification: 20D15, 20D45.
keywords and phrases: Finite p-group, Automorphism group.

(ii) $\langle x,y|x^p = y^p = z^p = 1, [x,y] = z, [x,z] = [y,z] = 1\rangle$, *where p is odd.*

Lemma 2.3. *Let G be a non-abelian p-group with cyclic Frattini subgroup. Assume that either $p > 2$ or $cl(G) = 2$. Then $|\mathrm{Aut}^{\Phi}(G)| = |G|/p$ and $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = |Z(G)|/p$.*

Theorem 2.4. *Let G be a finite non-abelian p-group with cyclic Frattini subgroup. Assume that either $p > 2$ or $cl(G) = 2$. Then $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = p$ if and only if G has one of the following types: $E * (\mathbb{Z}_p \times \mathbb{Z}_p)$, $E * \mathbb{Z}_{p^2}$ or $G_1 * \ldots * G_s$, $(s > 0)$ where E is an extraspecial p-group and $G_i \cong \langle x,y|x^{p^3} = y^p = 1, x^y = x^{1+p^2}\rangle$, for $i = 1,\ldots,s$.*

In what follows we consider the case $p = 2$. Before proceeding further we list two families of finite 2-groups introduced by Berger, Kovacs and Newman in [1].

$$D^+_{2^{n+3}} = \langle a,b,c|a^{2^{n+1}} = b^2 = c^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, [b,c] = 1\rangle,$$

$$Q^+_{2^{n+3}} = \langle a,b,c|a^{2^{n+1}} = b^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, a^{2^n} = c^2, [b,c] = 1\rangle,$$

both with $n > 1$.

Theorem 2.5. [1] *Let G be a finite purely non-abelian 2-group with cyclic Frattini subgroup. Then*

$$G = G_0 * G_1 * \ldots * G_s,$$

where $G_i \cong D_8$ for $i = 1,\ldots,s$, $|G_0| > 2$ if $s > 0$, and G_0 has one of the following types: cyclic, non-abelian with a cyclic maximal subgroup, namely $D_{2^n}, Q_{2^n}, S_{2^n}, M_{2^n}$ all with $n \geq 3$; and $D_{2^{n+2}} * \mathbb{Z}_4, S_{2^{n+2}} * \mathbb{Z}_4, D^+_{2^{n+3}}, Q^+_{2^{n+3}}, D^+_{2^{n+3}} * \mathbb{Z}_4$, all with $n > 1$. Conversely, every such group has cyclic Frattini subgroup.

Lemma 2.6. *Let G be one of the groups D_{2^n}, Q_{2^n} or S_{2^n}, all with $n \geq 3$. Then $\mathrm{Aut}^{\Phi}(G) \cong \mathrm{Inn}(G) \rtimes \mathbb{Z}_{2^{n-3}}$. In particular, if $n \geq 5$ then $|\mathrm{Aut}^{\Phi}_Z(G) : \mathrm{Inn}(G)| > 2$.*

Lemma 2.7. *Let G be one of the groups $D^+_{2^{n+3}}$, $Q^+_{2^{n+3}}$, $D_{2^{n+2}} * \mathbb{Z}_4, S_{2^{n+2}} * \mathbb{Z}_4$ or $D^+_{2^{n+3}} * \mathbb{Z}_4$, all with $n \geq 3$. Then $|\mathrm{Aut}^{\Phi}_Z(G) : \mathrm{Inn}(G)| > 2$.*

Lemma 2.8. *Let G be a non-abelian 2-group with cyclic Frattini subgroup and $cl(G) > 2$ such that $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = 2$. Then $Z(G) \leq \Phi(G)$ and so G is purely non-abelian group.*

From now on we shall suppose throughout that G is a finite non-abelian 2-group whose Frattini subgroup is cyclic. For the rest of the paper, we will make use of the notation of Theorem 2.5 without further mention. For simplicity, we let E denote the central product of s copies of the dihedral group D_8.

Lemma 2.9. *If G has one of the following types: $D_{16} * \mathbb{Z}_4, S_{16} * \mathbb{Z}_4$, $D^+_{32} * \mathbb{Z}_4$, $D_{16} * \mathbb{Z}_4 * E$, $S_{16} * \mathbb{Z}_4 * E$ or $D^+_{32} * \mathbb{Z}_4 * E$, then $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| > 2$.*

Proposition 2.10. *Assume that $s = 0$. If $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = 2$ and $cl(G) > 2$ then G has one of the following types: $D_{16}, Q_{16}, S_{16}, D^+_{32}$ or Q^+_{32}.*

Proposition 2.11. *Assume that $s > 0$. If $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = 2$ and $cl(G) > 2$ then G has one of the following types: $D_{16} * E, Q_{16} * E, S_{16} * E, D^+_{32} * E$ or $Q^+_{32} * E$.*

Theorem 2.12. *If G is one of the groups $D_{16}, Q_{16}, S_{16}, D^+_{32}, Q^+_{32}, D_{16} * E, Q_{16} * E, S_{16} * E, D^+_{32} * E$ or $Q^+_{32} * E$, then $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = 2$.*

From now, we give a result which deals with the occurrence of elementary abelian group in $\mathrm{Aut}^{\Phi}(G)$, where G is a finite p-group.

Lemma 2.13. *Let G be a finite group with $\Phi(G) \leq Z(G)$, $\Phi(G) = Dr \prod_{i=1}^{t} H_i$ and $G/G' = Dr\prod_{j=1}^{s}(K_j/G')$. If*

$$A_{ij} = \{\alpha_f | f \in Hom(K_i/G', H_j)\} \qquad (1 \leq i \leq s, 1 \leq j \leq t),$$

then

(i) $|A_{ij}| = |Hom(K_i/G', H_j)|$, *and* $|\mathrm{Aut}^{\Phi}(G)| = \prod_{i,j} |A_{ij}|$,
(ii) $\mathrm{Aut}^{\Phi}(G) = \prod_{j,i} A_{ij}$,
(iii) $\mathrm{Aut}^{\Phi}(G)$ *is abelian if and only if* $[A_{ij}, A_{kl}] = 1$ *for all* i, j, k, l.

Theorem 2.14. *Let G be a finite abelian 2-group. Then*

(i) *If G is cyclic, then* $\mathrm{Aut}^{\Phi}(G)$ *is elementary abelian if and only if* $G \cong \mathbb{Z}_4$ *or* $G \cong \mathbb{Z}_8$.
(ii) *If G is non-cyclic, then* $\mathrm{Aut}^{\Phi}(G)$ *is elementary abelian if and only if* $\exp(\Phi(G)) = 2$ *or* $G \cong \mathbb{Z}_8 \times H$, *where H is elementary abelian group.*

Lemma 2.15. *Let G be an abelian p-group, p odd, then* $\mathrm{Aut}^{\Phi}(G)$ *is elementary abelian if and only if* $\exp(\Phi(G)) = p$.

Lemma 2.16. *Let G be a finite non-abelian p-group, p odd, then* $\mathrm{Aut}^{\Phi}(G)$ *is elementary abelian if and only if* $\Phi(G) \leq Z(G)$ *and* $\exp(G/G') = p$ *or* $\exp(\Phi(G)) = p$.

Definition. Let G be a finite abelian p-group of type $(p^n, p, ..., p)$, where $n > 1$. If $G = A \times B$ is a direct decomposition G with A being cyclic of order p^n and B being elementary abelian, we call A and B a *cyclic part* and an *elementary part* of G, respectively. Such a group G is said to be a *ce-group*.

Theorem 2.17. *Let G be a finite non-abelian 2-group with non-cyclic Frattini subgroup. Then* $\mathrm{Aut}^{\Phi}(G)$ *is elementary abelian if and only if* $\Phi(G) \leq Z(G)$ *and one of the following conditions holds:*

(i) $\exp(G/G') = 2$ *or* $\exp(\Phi(G)) = 2$*;*
(ii) $\exp(\Phi(G)) = 4$ *and* G/G', $\Phi(G)$ *are ce-groups having the property that an elementary part of* $\Phi(G)$ *is contained in* G' *of index at most* 2. *Moreover if an elementary part of* $\Phi(G)$ *is equal to* G', *then* $\exp(G/G') = 8$, *otherwise,* $\exp(G/G') = 4$.

Proposition 2.18. *Let G be a finite non-abelian 2-group with cyclic Frattini subgroup. Then* $\mathrm{Aut}^{\Phi}(G)$ *is elementary abelian if and only if* $\Phi(G) \leq Z(G)$ *and one of the following conditions holds:*

(i) $\exp(G/G') = 2$ *or* $\exp(\Phi(G)) = 2$*;*
(ii) $\Phi(G) \cong \mathbb{Z}_4$ *and* $G/G' \cong \mathbb{Z}_4 \times H$, *where H is elementary abelian group.*

References

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