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GENERALIZATIONS OF D_{11} AND D_{11}^{+} MODULES

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Abstract. Let M be a module. Then M is called a D_{11} module if any submodule of M has a supplement which is a direct summand of M. Also M is called D_{11}^{+} if every direct summand of M is D_{11}. In this paper we investigate generalizations of D_{11} and D_{11}^{+} modules, namely δ–D_{11} and δ–D_{11}^{+} modules. We will prove that any δ–D_{11} module M has a decomposition $M = M_1 \oplus M_2$ with $\delta(M_1) \ll_\delta M_1$ and $\delta(M_2) = M_2$.

1. Introduction

Throughout this article, all rings R are associative and have an identity, and all modules are unitary right R-modules.

We write $N \leq M$ to denote that N is a submodule of the module M while $N \subseteq^{\oplus} M$ means that N is a direct summand of M. A submodule L of M is called *small* in M (denoted by $L \ll M$) if, for every proper submodule K of M, $L + K \neq M$. A submodule N of M is called *essential* in M (denoted by $N \trianglelefteq_e M$) if $N \cap K \neq 0$ for every nonzero submodule K of M. The *singular* submodule of a module M (denoted by $Z(M)$) is $Z(M) = \{x \in M \mid Ix = 0$ for some ideal $I \trianglelefteq_e R\}$. A module M is called *singular* (*nonsingular*) if $Z(M) = M$ (resply. $Z(M) = 0$).

We denote the ring of all endomorphisms of M by $End(M)$ and the Jacobson radical of M by $Rad(M)$. A submodule N of M is called *fully invariant* in M if $f(N) \leq N$ for every $f \in End(M)$.

A module M is called *lifting* (or said to *satisfy condition* D_1) if, for every submodule N of M, M has a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll B$. M is said to *satisfy condition* D_3 if, whenever A and B are direct summands of M with $M = A + B$, then $A \cap B$ is also a direct summand of M.

For two submodules N and K of the module M, N is called a *supplement* of K in M if N is minimal with respect to the property $M = K + N$, equivalently $M = K + N$ and $N \cap K \ll N$. The module M is called *supplemented* if every submodule of M has a supplement in M. The module M is called a D_{11} module if every submodule of M has a supplement that is a direct summand

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of M. Furthermore, M is called a D_{11}^{+} module if each of its direct summands is D_{11}.

δ–small submodules were defined as a generalization of small submodules by Zhou in [4]. Let M be a module and $L \leq M$. Then L is called *δ–small* in M (denoted by $L \ll_\delta M$) if, for any submodule N of M with M/N singular, $M = N + L$ implies that $M = N$. The sum of all δ–small submodules of M is denoted by $\delta(M)$.

The module M is called *δ–lifting* (or we say that *M has δ–D_1*) if, for any submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll_\delta M_2$.

In this paper we investigate generalizations of D_{11} and D_{11}^{+} modules, namely δ–D_{11} modules and δ–D_{11}^{+} modules. A module M is called a δ–D_{11} module if every submodule of M has a δ–supplement that is a direct summand of M. Also M is called a δ–D_{11}^{+} module if every direct summand of M is δ–D_{11}.

Let M be any module and $B \leq A$ be submodules of M. Then B is called a *δ–cosmall* submodule of A in M if $A/B \ll_\delta M/B$. A submodule N of M is called *δ–coclosed* in M if N has no proper δ–cosmall submodule in M, that is, if $B \leq N$ such that $N/B \ll_\delta M/B$, then $N = B$. A submodule A of M is *weak δ–coclosed* in M if, given $B \leq A$ such that A/B is singular and $A/B \ll_\delta M/B$, then $A = B$. For a submodule N of M, $A \leq N$ is called a *δ–coclosure* of N in M if A is δ–coclosed in M and $N/A \ll_\delta M/A$ and A is called a *weak δ–coclosure* of N in M if A is weak δ–coclosed in M and $N/A \ll_\delta M/A$.

2. δ–D_{11} modules

Recall that a module M is called a δ–D_{11} module if any submodule of M has a δ–supplement which is a direct summand of M.

Clearly any D_{11} module is δ–D_{11} and any δ–D_{11} module is δ–supplemented.

Lemma 2.1. *Let M be a module and $N \subseteq^{\oplus} M$. Then N is weak δ–coclosed in M.*

Lemma 2.2. *Let M be a module and $A \leq N \leq M$ such that N is weak δ–coclosed in M. Then $A \ll_\delta M$ implies that $A \ll_\delta N$.*

Lemma 2.3. *Let M be a module and $A \leq N \leq M$ be such that N is δ–coclosed in M. Then $A \ll_\delta M$ implies that $A \ll N$.*

A module M is said to have the *summand intersection property* if the intersection of two summands of M is again a summand of M.

Theorem 2.4. *Let M be a δ–D_{11} module and N be a weak δ–coclosed submodule of M. If the intersection of N with any direct summand of M is a direct summand of N, then N has δ–D_{11}. In particular, if M has the summand intersection property, then every direct summand of M has δ–D_{11}.*

Corollary 2.5. *Let M be a δ–D_{11} module and N be a δ–coclosed submodule of M. If the intersection of N with any direct summand of M is a direct summand of N, then N has D_{11}.*

Corollary 2.6. *Let M be a δ–D_{11} module and N be a weak δ–coclosed (δ–coclosed) submodule of M. If $eN \leq N$ for all $e = e^2 \in End(M)$, then N has δ–D_{11} (D_{11}). In particular any fully invariant δ–coclosed submodule of M has D_{11}.*

Theorem 2.7. *Any finite direct sum of δ–D_{11} modules is a δ–D_{11} module.*

Theorem 2.8. *Any δ–D_{11} module M has a decomposition $M = M_1 \oplus M_2$, where $\delta(M_1) \ll_\delta M_1$ and $\delta(M_2) = M_2$.*

The submodule $Z^*(M)$ of M is defined by $Z^*(M) = \{m \in M : mR$ is small in $E(mR)\}$ where $E(mR)$ is the injective hull of mR (see for example [1]). We define the submodule $\delta^*(M)$ of M by $\delta^*(M) = \{m \in M : mR$ is δ–small in $E(mR)\}$. Note that if $M = M_1 \oplus M_2$ we have $Z^*(M) = Z^*(M_1) \oplus Z^*(M_2)$ and $\delta^*(M) = \delta^*(M_1) \oplus \delta^*(M_2)$.

Theorem 2.9. *Let M be a δ–D_{11} module. Then there exists a decomposition $M = M_1 \oplus M_2$, where $\delta^*(M_1) \ll_\delta M_1$ and $\delta^*(M_2) = M_2$.*

Lemma 2.10. *Let M be a nonzero module and N a fully invariant submodule of M. If $M = M_1 \oplus M_2$, then $N = (N \cap M_1) \oplus (N \cap M_2)$.*

Theorem 2.11. *Let N be a fully invariant submodule of the module M. If M has δ–D_{11}, then M/N has δ–D_{11}. Moreover if N is a direct summand of M, then N has also δ–D_{11}.*

Example 2.12. (1) Let M denote the $\mathbb{Z}$-module $\mathbb{Z}/6\mathbb{Z}$. The proper nonzero submodules of M are $2\mathbb{Z}/6\mathbb{Z}$ and $3\mathbb{Z}/6\mathbb{Z}$ and we have $\mathbb{Z}/6\mathbb{Z} = 2\mathbb{Z}/6\mathbb{Z} \oplus 3\mathbb{Z}/6\mathbb{Z}$. Hence M is a δ–D_{11} module.
(2) The $\mathbb{Z}$-module $\mathbb{Z}$ is not a δ–D_{11} module.

3. δ–D_{11}^{+} modules

Recall that a module M is called a δ–D_{11}^{+} module if every direct summand of M is δ–D_{11}.

By [2, Lemma 2.3], any direct summand of a δ-lifting module is δ–lifting, so any distributive δ-lifting module is δ–D_{11}^{+}.

A module M is said to have the *finite exchange property* if, for any finite index set I, whenever $M \oplus N = \oplus_{i\in I} A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\oplus_{i\in I} B_i)$ for submodules $B_i \leq A_i$.

Lemma 3.1. *The finite exchange property is inherited by summands and finite direct sums.*

Theorem 3.2. *Let $\{N_i\}_{i=1}^{n}$ be a family of modules with δ–D_{11}^{+} and the finite exchange property. Then $\oplus_{i=1}^{n} N_i$ has δ–D_{11}^{+}.*

Example 3.3. Let M denote the $\mathbb{Z}$–module $\mathbb{Z}/a^i\mathbb{Z} \oplus \mathbb{Z}/a^j\mathbb{Z}$, where a is a nonzero integer and $i, j \in \mathbb{N}$. Then M is a module with δ–D_{11}^{+} (see [1, Example 2.16]).

Theorem 3.4. *Let M be a module with δ–D_{11} and D_3. Then M has δ–D_{11}^{+}.*

Let M be a module and $N \leq M$. Then $A \leq M$ is called an *essential extension* of N if $N \trianglelefteq A$. A submodule N is called *closed* in M if N has no proper essential extension in M. A submodule B of M is called a *closure* of N in M if B is closed and an essential extension of N in M.

P. F. Smith [3] calls a module M a *UC-module* if every submodule of M has a unique closure in M. Also M is called *extending* if every closed submodule of M is a direct summand of M.

Lemma 3.5. *Any UC extending module has* D_3.

Theorem 3.6. *A UC extending module* M *is* δ–D_{11} *if and only if it is* δ–D_{11}^+.

Theorem 3.7. *Let* M *be a module with* D_3. *Then the following conditions are equivalent :*
(1) M *has* δ–D_{11}^+.
(2) M *has* δ–D_{11}.
(3) $M = M_1 \oplus M_2$, *where both* M_1 *and* M_2 *are* δ–D_{11}, $\delta(M_1) \ll_\delta M_1$, *and* $\delta(M_2) = M_2$.
(4) $M = M_1 \oplus M_2$, *where both* M_1 *and* M_2 *have* δ–D_{11}, $\delta^*(M_1) \ll_\delta M_1$, *and* $\delta^*(M_2) = M_2$.

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