Tarbiat Moallem University, 20^{th} Seminar on Algebra,
2-3 Ordibehesht, 1388 (Apr. 22-23, 2009) pp 219-221

DUAL STABILIZERS AND DUAL NORMAL *BCK*-ALGEBRAS

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ABSTRACT. In this note, the notions of dual right and dual left stabilizers of a set in bounded *BCK*-algebras are defined and the relationship between of them are investigated. Then we define a class of special bounded *BCK*-algebras called dual normal *BCK*-algebras. Finally we prove that the dual semisimple bounded *BCK*-algebras and dual *J*-semisimple bounded *BCK*-algebras are all dual normals.

1. INTRODUCTION AND PRELIMINARIES

The study of *BCK*-algebras was initiated by Y. Imai and K-Iseki [1] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Dual ideals are important in bounded *BCK*-algebras. In 1986, the notion of dual ideals in bounded *BCK*-algebras was introduced by J. Meng [4] and gave certain properties of it. Now, in this paper we define the dual left and dual right stabilizers and dual normal *BCK*-algebras, as mentioned in the abstract.

We give herein the basic notions on *BCK*-algebras. For further information, we refer to the book [3]. By a *BCK*-algebra we mean an algebra $(X,*,0)$ of type (2,0) satisfying the following axioms: for every $x,y,z\in X$,
(i) $((x*y)*(x*z))*(z*y)=0$,
(ii) $(x*(x*y))*y=0$,
(iii) $x*x=0$,
(iv) $x*y=y*x=0\Rightarrow x=y$,
(v) $0*x=0$.

We can define a partial ordering $\leq$ by $x\leq y$ if and only if $x*y=0$. A *BCK*-algebra X is said to be *commutative* if $x*(x*y)=y*(y*x)$, for all $x,y\in X$.

If there is an element 1 of X satisfying $x\leq 1$, for all $x\in X$, then the element 1 is called *unit* of X. A *BCK*-algebra with unit is called *bounded* . In a bounded *BCK*-algebra, we denote $1*x$ by Nx and $NA=\{Nx\in X|\ \ x\in A\}$, for all $A\subseteq X$.

A bounded *BCK*-algebra X is called *involutory* if $NNx=x$, for all $x\in X$.

A nonempty subset D of a bounded *BCK*-algebra X is called a *dual ideal* if
(i) $1\in D$, (ii) $N(Nx*Ny)\in D$ and $y\in D$ imply that $x\in D$, for any $x,y\in X$.

Given a bounded *BCK*-algebra X and a nonempty subset A of X. The intersection of all dual ideals containing A is called the dual ideal generated by A, written $[A]$. It is proved that $[A]=\{x\in X:(...(Nx*Na_1)*...)*Na_n=0, for some\ \ a_1,a_2,...,a_n\in A\}$. Let A be a nonempty subset of *BCK*-algebra X. Then the sets $A_l^*=\{x\in X|\ \ a*x=a,\forall a\in A\}$

and $A_r^* = \{x \in X| \ \ x * a = x, \forall a \in A\}$ are called the left and right stabilizers of A, respectively and the set $A^* = A_l^* \cap A_r^*$ is called the stabilizer of A.

2. Dual stabilizers

In the sequel let X be a bounded BCK-algebra with unit 1.

Definition 2.1. Let A be a nonempty subset of X. Then the sets

$$DA_l = \{x \in X| \ \ Na * Nx = Na, \forall a \in A\}$$

and

$$DA_r = \{x \in X| \ \ Nx * Na = Nx, \forall a \in A\}$$

are called the dual left and dual right stabilizers of A, respectively and the set $DA = DA_l \cap DA_r$ is called the dual stabilizer of A.
For convenience the dual stabilizer, dual left and dual right stabilizers of a single element set $A = \{a\}$ are denoted by DS_a, DL_a and DR_a, respectively.

Theorem 2.2. *Let X be an involutory BCK-algebra and A be a nonempty subset of X. Then:*
(i) $N(DA_l) = (NA)_l^*$, $N(DA_r) = (NA)_r^*$ *and* $N(DA) = (NA)^*$,
(ii) $N(A_l^*) = D(NA)_l$, $N(A_r^*) = D(NA)_r$ *and* $N(A^*) = D(NA)$.

Theorem 2.3. *Let A be a nonempty subset of X. Then DA_l is a dual ideal of X.*

The following example shows that DA_r is not a dual ideal in general.

Example 2.4. Let $X = \{0, 1, 2, 3, 4\}$ and let $*$ operation be given by the following table

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Then $(X, *, 0)$ is a bounded BCK-algebra with unit 4. Consider $A = \{3\}$, it is easy to check that $DA_r = \{1, 4\}$. Now we see that $4 * ((4 * 3) * (4 * 1)) = 4 \in DA_r$ and $1 \in DA_r$ but $3 \notin DA_r$. Hence DA_r is not a dual ideal of X.

Theorem 2.5. *Let X be a bounded commutative BCK-algebra and A be a nonempty subset of X. Then $DA_r = DA_l = DA$ and so DA_r is a dual ideal of X.*

Theorem 2.6. *Let A be a nonempty subset of X. Then*
(i) $[A] \cap DA_r = \{1\}$,
(ii) $DA_r = D[A]_r$,
(iii) *If DA_r is a dual ideal of X, then $DA = DA_r$.*

Theorem 2.7. *Let A and B be nonempty subsets of X. Then*
(i) $A \cap DA_l = \emptyset$ *or* $\{1\}$, $A \cap DA_r = \emptyset$ *or* $\{1\}$ *and* $A \cap DA = \emptyset$ *or* $\{1\}$,
(ii) *if* $A \subseteq B$, *then* $DB_l \subseteq DA_l$, $DB_r \subseteq DA_r$ *and* $DB \subseteq DA$,
(iii) $A \subseteq D(DA_r)_l \cap D(DA_l)_r$ *and* $A \subseteq D(DA)$,
(iv) $DA_l = D(D(DA_l)_r)_l$, $DA_r = D(D(DA_r)_l)_r$ *and* $DA = D(D(DA))$,

(v) $D(A\cup B)_l = DA_l \cap DA_l$, $D(A\cup B)_r = DA_r \cap DA_r$ *and* $D(A\cup B) = DA\cap DB$,
(vi) $DA_l = \bigcap_{a\in A} DL_a$, $DA_r = \bigcap_{a\in A} DR_a$ *and* $DA = \bigcap_{a\in A} DS_a$.

Theorem 2.8. *If A is a dual ideal of X, then DA is a dual ideal of X.*

Theorem 2.9. *Let A, B be two dual ideals of X. Then $A\cap B = \{1\}$ if and only if $A \subseteq DB$.*

Theorem 2.10. *Let A be a dual ideal of X. Then $DA_l = DA \subseteq DA_r$. In particular, if DA_r is a dual ideal of X, $DA = DA_l = DA_r$.*

3. DUAL NORMAL *BCK*-ALGEBRAS

Definition 3.1. A bounded BCK-algebra X is called dual normal BCK-algebra, if the dual right stabilizer DR_a of any element $a\in X$ is a dual ideal of X.

Note that any commutative bounded BCK-algebra is dual normal BCK-algebra.

Theorem 3.2. *The following statements are equivalent:*
(i) *X is dual normal,*
(ii) $DR_a \subseteq DL_a$, $\forall a\in A$,
(iii) $DR_a = DL_a$, $\forall a\in A$,
(v) $Nx * Ny = Nx$ *implies* $Ny * Nx = Ny$, $\forall x, y\in X$.

Definition 3.3. A bounded BCK -algebra X is called dual semisimple if every dual ideal A of X is a sub-summand of X, i.e. there exists dual ideal B of X such that $A\bigcap B = \{1\}$ and $X = [A\bigcup B]$.

Theorem 3.4. *Every dual semisimple bounded BCK-algebra is dual normal .*

Definition 3.5. The dual J-radical , denoted by $DJ(X)$, of a bounded BCK-algebra X means the intersection of all maximal dual ideals of X. By Zorn's Lemma the collection of maximal dual ideals of X is nonempty. If $DJ(X) = \{1\}$, then X is called dual J-semisimple.

Theorem 3.6. *Every dual J-semisimple BCK-algebra is dual normal.*

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