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AN OPEN PROBLEM ON SEMIPRIME MODULES

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ABSTRACT. An R-module M is called semiprime if it is cogenerated by each of its essential submodules. We investigate when a semiprime module M has the property that for every non-zero submodule $N \leq M$, there exists an R-homomorphism $\theta : M \to N$ such that $\theta(N) \neq 0$. It is partially solved an open problem posed in [2].

1. INTRODUCTION

All rings are associative with unit elements and all modules are unitary right modules. It is easy to verify that a ring R is semiprime (i.e. R has no non-trivial nilpotent ideals) if and only if R is cogenerated by each of its essential right ideals (i.e. if A is an essential right ideal of R, there exists a family of R-modules $\{A_i\}_{i\in I}$ such that each A_i is isomorphic to A and R_R embeds in $\Pi_{i\in I}A_i$). Motivated by this, an R-module M is called *semiprime* if it is cogenerated by each of its essential submodules, see [4, pp. 100] for transferring semiprimeness conditions from rings to modules. A proper subclass of the class of semiprime modules is the class of modules X which embed in each of their essential submodules. Such modules X was called *essentially compressible* and studied in [3]. In [2, open problem(2), p.92], it is asked whether there exists a semiprime module M which is not *weakly compressible*. Recall that a module M_R is weakly compressible if for any non-zero submodule N of M_R there exists an R-endomorphism $\varphi : M \to N$ such that $\varphi|_N \neq 0$. Clearly weakly compressible modules M are *retractable* (i.e. $\mathrm{Hom}_R(M, N) \neq 0$ for every non-zero submodule N of M). Retractable modules have been investigated by several authors (see, [1] for a recent work on the subject). In order to study semiprime modules by prime modules, there exists the natural question:
" *when is a semiprime module a subdirect product of prime modules*? "
In [2], it is proved that every weakly compressible module is a subdirect product of prime modules where a module M_R is said to be prime whenever $R/\mathrm{ann}(M) \in \mathrm{Cog}(N)$ for all $N \leq M_R$. In this note, in section 1, we investigate conditions under which a semiprime module is weakly compressible and in section 2, we show that over certain rings, semiprime modules are weakly compressible. We

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introduce and investigate two classes of modules such that the class of weakly compressible modules lies between them. In the above definition of weakly compressible modules, suppose that N is an essential submodule of M, then we call M *essentially weakly compressible*. We also introduce *semi-essentially compressible* modules X (i.e. $Y \leq_e X_R$ implies $X \hookrightarrow Y^{(\Lambda)}$ for some set Λ). If a semiprime module X has the property that $Y \leq_e X$ when $X \in \mathrm{Cog}(Y)$, then we call X *good semiprime*. Modules whose submodules are fully invariant are called *duo*. Any terminology not defined here may be found in [4].

2. Main results

In the following Lemma we observe that semiprime modules are (at least) essentially weakly compressible.

Lemma 2.1. (i) *The class of weakly compressible modules is closed under direct sums and taking submodules.*
(ii) *Any product of weakly compressible modules is an essentially weakly compressible.*
(iii) *Semiprime modules are essentially weakly compressible.*

Using the known fact that torsionless modules over semiprime rings are weakly compressible, we can prove that nonsingular semiprime modules are weakly compressible.

Theorem 2.2. *Nonsingular semiprime modules are weakly compressible.*

It is easy to verify that if R is a duo ring, then every right ideal with zero annihilator is an essential right ideal. Next result extends this to modules.

Theorem 2.3. *Good semiprime modules are weakly compressible.*

Following Lemma give examples of good semiprime modules.

Lemma 2.4. *Duo semiprime modules are good semiprime.*

Corollary 2.5. *Duo semiprime modules are weakly compressible.*

By lemma 2.1(i) and the fact that any product of simple modules is a weakly compressible, we have:

Theorem 2.6. *If* $Soc(M) \leq_e M$ *and* M *is semiprime, then* M *is weakly compressible.*

We now investigate semi-essentially compressible modules and give a characterization of them and show that they are weakly compressible. This extends [3, Theorem 3.1]

Theorem 2.7. *Consider the following conditions for* M_R.
(a) M *is a semi-essentially compressible* R*-module.*
(b) $\forall\ N \leq M,\ \exists\ \alpha : M_R \to N_R^{(I)}$ *for some set* I *such that* $N \cap \ker \alpha = 0$.
(c) $\hat{M}_R$ *has no proper fully invariant essential submodules.*
Then (a)⇔(b)⇒(c) and all conditions are equivalent if M_R *is* $\sum$*-projective.*

Corollary 2.8. *Semi-essentially compressible modules are weakly compressible.*

3. Weakly compressible modules over certain rings

By Theorem 2.6, we give a positive answer to the open problem when R is a semi-Artinian ring.

Theorem 3.1. *If* R *is a semi-Artinian ring, then every semiprime module is weakly compressible.*

Since the class of semiprime modules is closed under taking essential submodules, we can conclude the following by Lemma 2.1.

Theorem 3.2. *If* R *is a commutative full semiprime ring, then every semiprime module lies in a product of weakly compressible modules.*

Finally, we give a criteria for the weakly compressibility of a semiprime module over a ring with Krull dimension, in particular over Noetherian rings.

Theorem 3.3. *If* R *is a commutative ring with Krull dimension, then a semiprime* R*-module* M *is weakly compressible if and only if there exists* $f : M \to C$ *with* $f(C) \neq 0$ *whenever* $C \leq M_R$ *is a critical compressible submodule whose Krull dimension is minimal among all Krull dimensions of critical compressible submodules.*

The following conjecture states that the study of semiprime modules which are weakly compressible reduces to the study of such modules when they are singular.

Conjecture. *A semiprime module is a weakly compressible module if and only if its singular submodule is a weakly compressible module.*

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