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# AN OPEN PROBLEM ON SEMIPRIME MODULES

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Abstract. An $R$-module $M$ is called semiprime if it is cogenerated by each of its essential submodules. We investigate when a semiprime module $M$ has the property that for every non-zero submodule $N \leq M$, there exists an $R$-homomorphism $\theta : M \to N$ such that $\theta(N) \neq 0$. It is partially solved an open problem posed in [2].

## 1. Introduction

All rings are associative with unit elements and all modules are unitary right modules. It is easy to verify that a ring $R$ is semiprime (i.e. $R$ has no non-trivial nilpotent ideals) if and only if $R$ is cogenerated by each of its essential right ideals (i.e. if $A$ is an essential right ideal of $R$, there exists a family of $R$-modules $\{A_i\}_{i\in I}$ such that each $A_i$ is isomorphic to $A$ and $R_R$ embeds in $\Pi_{i\in I} A_i$). Motivated by this, an $R$-module $M$ is called *semiprime* if it is cogenerated by each of its essential submodules, see [4, pp. 100] for transferring semiprimeness conditions from rings to modules. A proper subclass of the class of semiprime modules is the class of modules $X$ which embed in each of their essential submodules. Such modules $X$ was called *essentially compressible* and studied in [3]. In [2, open problem(2), p.92], it is asked whether there exists a semiprime module $M$ which is not *weakly compressible*. Recall that a module $M_R$ is weakly compressible if for any non-zero submodule $N$ of $M_R$ there exists an $R$-endomorphism $\varphi : M \to N$ such that $\varphi|_N \neq 0$. Clearly weakly compressible modules $M$ are *retractable* (i.e. $\mathrm{Hom}_R(M, N) \neq 0$ for every non-zero submodule $N$ of $M$). Retractable modules have been investigated by several authors (see, [1] for a recent work on the subject). In order to study semiprime modules by prime modules, there exists the natural question:
" *when is a semiprime module a subdirect product of prime modules*? "
In [2], it is proved that every weakly compressible module is a subdirect product of prime modules where a module $M_R$ is said to be prime whenever $R/\mathrm{ann}(M) \in \mathrm{Cog}(N)$ for all $N \leq M_R$. In this note, in section 1, we investigate conditions under which a semiprime module is weakly compressible and in section 2, we show that over certain rings, semiprime modules are weakly compressible. We

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introduce and investigate two classes of modules such that the class of weakly compressible modules lies between them. In the above definition of weakly compressible modules, suppose that $N$ is an essential submodule of $M$, then we call $M$ *essentially weakly compressible*. We also introduce *semi-essentially compressible* modules $X$ (i.e. $Y \leq_e X_R$ implies $X \hookrightarrow Y^{(\Lambda)}$ for some set $\Lambda$). If a semiprime module $X$ has the property that $Y \leq_e X$ when $X \in \mathrm{Cog}(Y)$, then we call $X$ *good semiprime*. Modules whose submodules are fully invariant are called *duo*. Any terminology not defined here may be found in [4].

## 2. Main results

In the following Lemma we observe that semiprime modules are (at least) essentially weakly compressible.

**Lemma 2.1.** (i) *The class of weakly compressible modules is closed under direct sums and taking submodules.*
(ii) *Any product of weakly compressible modules is an essentially weakly compressible.*
(iii) *Semiprime modules are essentially weakly compressible.*

Using the known fact that torsionless modules over semiprime rings are weakly compressible, we can prove that nonsingular semiprime modules are weakly compressible.

**Theorem 2.2.** *Nonsingular semiprime modules are weakly compressible.*

It is easy to verify that if $R$ is a duo ring, then every right ideal with zero annihilator is an essential right ideal. Next result extends this to modules.

**Theorem 2.3.** *Good semiprime modules are weakly compressible.*

Following Lemma give examples of good semiprime modules.

**Lemma 2.4.** *Duo semiprime modules are good semiprime.*

**Corollary 2.5.** *Duo semiprime modules are weakly compressible.*

By lemma 2.1(i) and the fact that any product of simple modules is a weakly compressible, we have:

**Theorem 2.6.** *If* $Soc(M) \leq_e M$ *and* $M$ *is semiprime, then* $M$ *is weakly compressible.*

We now investigate semi-essentially compressible modules and give a characterization of them and show that they are weakly compressible. This extends [3, Theorem 3.1]

**Theorem 2.7.** *Consider the following conditions for* $M_R$.
(a) $M$ *is a semi-essentially compressible* $R$*-module.*
(b) $\forall\ N \leq M$, $\exists\ \alpha : M_R \to N_R^{(I)}$ *for some set* $I$ *such that* $N \cap \ker \alpha = 0$.
(c) $\hat{M}_R$ *has no proper fully invariant essential submodules.*
*Then (a)⇔(b)⇒(c) and all conditions are equivalent if* $M_R$ *is* $\sum$*-projective.*

**Corollary 2.8.** *Semi-essentially compressible modules are weakly compressible.*

## 3. Weakly compressible modules over certain rings

By Theorem 2.6, we give a positive answer to the open problem when $R$ is a semi-Artinian ring.

**Theorem 3.1.** *If* $R$ *is a semi-Artinian ring, then every semiprime module is weakly compressible.*

Since the class of semiprime modules is closed under taking essential submodules, we can conclude the following by Lemma 2.1.

**Theorem 3.2.** *If* $R$ *is a commutative full semiprime ring, then every semiprime module lies in a product of weakly compressible modules.*

Finally, we give a criteria for the weakly compressibility of a semiprime module over a ring with Krull dimension, in particular over Noetherian rings.

**Theorem 3.3.** *If* $R$ *is a commutative ring with Krull dimension, then a semiprime* $R$*-module* $M$ *is weakly compressible if and only if there exists* $f : M \to C$ *with* $f(C) \neq 0$ *whenever* $C \leq M_R$ *is a critical compressible submodule whose Krull dimension is minimal among all Krull dimensions of critical compressible submodules.*

The following conjecture states that the study of semiprime modules which are weakly compressible reduces to the study of such modules when they are singular.

**Conjecture.** *A semiprime module is a weakly compressible module if and only if its singular submodule is a weakly compressible module.*

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