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## SOME ANNIHILATOR CONDITIONS IN COMMUTATIVE RINGS

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**ABSTRACT.** Let  $R$  be a commutative ring and let  $M$  be an  $R$ -module. Denote by  $Z_R(M)$  the set of all zero-divisors of  $R$  on  $M$ .  $M$  is called *strongly primal* (resp. *super primal*) if for arbitrary  $a, b \in Z_R(M)$  (resp. every finite subset  $F$  of  $Z_R(M)$ ) the annihilator of  $\{a, b\}$  (resp.  $F$ ) in  $M$  is non-zero. In this paper we give some results on these classes of modules. Also we provide a relationship among the families of primal, strongly primal and super primal modules.

### 1. INTRODUCTION

Throughout this paper all rings are commutative with nonzero identity, and all modules are considered to be unitary.

For a given ring  $R$ , an  $R$ -module  $M$  and a submodule  $N$  of  $M$ , we will denote by  $(N :_R M)$  the residual of  $N$  by  $M$ , the set of all  $r$  in  $R$  such that  $rM \subseteq N$ . The annihilator of  $M$ , denoted by  $\text{ann}_R(M)$ , is  $(0 :_R M)$ . For every subset  $S$  of  $R$ , we denote by  $\text{Ann}_M(S)$  the set of elements  $m \in M$  such that  $ma = 0$  for each  $a \in S$ . An element  $r \in R$  is called a zero-divisor on  $M$  provided that there exists  $0 \neq m \in M$  such that  $rm = 0$ , that is  $\text{Ann}_M(r) \neq 0$ . We denote by  $Z_R(M)$  the set of all zero-divisors of  $R$  on  $M$ .

An annihilator condition on a commutative ring  $R$  is property (A).  $R$  is said to have property (A) if every finitely generated ideal  $I$  contained in  $Z(R)$  has a nonzero annihilator ([1]). A ring with property (A) is called a McCoy ring. An  $R$ -module  $M$  is said to be McCoy provided that for every finitely generated ideal  $I$  of  $R$  with  $I \subseteq Z_R(M)$ ,  $\text{Ann}_M(I) \neq 0$ .

Let  $R$  be a commutative ring,  $M$  an  $R$ -module and  $N$  a submodule of  $M$ . An element  $r \in R$  is called prime to  $N$  if  $rm \in N$  ( $m \in M$ ) implies that  $m \in N$ , that is  $(N :_M r) = \{m \in M : rm \in N\} = N$ . Denote by  $S(N)$  the set of all elements of  $R$  that are not prime to  $N$ . Then  $N$  is said to be *primal* if  $S(N)$  forms an ideal; this ideal is always a prime ideal, called the adjoint ideal  $P$  of  $N$ . In this case we also say that  $N$  is a  $P$ -primal submodule of  $M$ . If the zero submodule of  $M$  is primal, then  $M$  will be called a primal module. It

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is clear that  $S(0) = Z_R(M)$ . Hence  $M$  is a primal  $R$ -module if  $Z_R(M)$  forms an ideal of  $R$ . The ring  $R$  is primal if it is primal as an  $R$ -module. It is easy to check that a submodule  $N$  of an  $R$ -module  $M$  is *primal* if and only if the factor module  $M/N$  is primal as an  $R/(N :_R M)$ -module. The  $R$ -module  $M$  is called *strongly primal* (resp. *super primal*) if for arbitrary  $a, b \in Z_R(M)$  (resp. every finite subset  $F$  of  $Z_R(M)$ ) the annihilator of  $\{a, b\}$  (resp.  $F$ ) in  $M$  is non-zero. Clearly,  $M$  is a super primal  $R$ -module if and only if  $M$  is a primal and McCoy.

The submodule  $N$  of the  $R$ -module  $M$  is called *strongly primal* (resp. *super primal*) if  $M/N$  is a strongly primal (super primal)  $R/(N :_R M)$ -module.

It is clear that every super primal module is strongly primal and every strongly primal module is primal. It is shown in Examples 1.1 that the converse implications do not hold.

**Example 1.1.** (1) A McCoy  $R$ -module need not be strongly McCoy. For example, let  $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ . Then  $R$  is a McCoy ring which is not strongly McCoy.  
 (2) Let  $R = \mathbb{Z}$  and consider the  $R$ -module  $M = \mathbb{Z}_2 \times \mathbb{Z}_3$ . Then  $M$  is not a primal  $R$ -module, while it is McCoy.  
 (3) In this example we use the concept of so-called  $A + B$ -rings introduced in [3]. Let  $K$  be a field,  $w, y$  and  $z$  algebraically independent indeterminates,  $M = (w, y, z)K[w, y, z]$ , and let  $D = K[w, y, z]_M$ . Clearly  $D$  is a local ring. Let  $Q$  be the maximal ideal of  $D$  and let  $P$  denote the set of height two primes of  $D$ . For each  $P_\alpha \in P$ , let  $Q_\alpha = Q/P_\alpha$ . Let  $I = A \times \mathbb{N}$  where  $A$  is an index set for  $P$  and let  $B = \sum Q_i$  where  $Q_i = Q_\alpha$  for each  $i = (\alpha, n) \in I$ . Set  $R = D + B$  the ring constructed from  $D \times B$  by setting  $(r, a) + (s, b) = (r+s, a+b)$  and  $(r, a)(s, b) = (rs, rb + sa + ab)$ . Then  $Z(R) = Q + B$  is a prime ideal of  $R$ . Hence  $R$  is a primal ring which is not McCoy..

**Proposition 1.2.** Let  $R$  be a commutative ring,  $M$  an  $R$ -module and  $N$  a submodule of  $M$ . Then  $f(x) \in R[x]$  is not prime to  $N[x]$  if and only if  $mf(x) \in N[x]$  for some  $m \in M \setminus N$ .

One can easily check that for every module  $M$  over a commutative ring  $R$ ,  $Z_R(M) \subseteq Z_{R[x]}(M[x])$  and it is easy to see that if  $M[x]$  is a primal  $R[x]$ -module then  $Z_{R[x]}(M[x]) = Z_R(M)[x]$ . Combining this result by Proposition 1.2, we get:

**Proposition 1.3.** Let  $R$  be a commutative ring and let  $M$  be an  $R$ -module. Then the following conditions are equivalent

- (i)  $M[x]$  is a primal  $R[x]$ -module;
- (ii)  $M$  is a super primal  $R$ -module;
- (iii)  $M[x]$  is a super primal  $R[x]$ -module;
- (iv)  $M[x]$  is a strongly primal  $R[x]$ -module.

Proposition 1.3 shows that if we can find the conditions under which  $M[x]$  is a primal  $R[x]$ -module, then one can have conditions under which the module

$M$  is super primal and hence primal. In the following results we give some conditions for which the module  $M[x]$  is primal.

**Proposition 1.4.** *Let  $R$  be a commutative ring,  $M$  an  $R$ -module and  $N$  an irreducible submodule of  $M$ . Then*

- (1)  $N[x]$  is a primal submodule of  $M[x]$ .
- (2) If  $P$  is the adjoint ideal of  $N$ , then  $P[x]$  is the adjoint ideal of  $N[x]$ .

**Theorem 1.5.** *Let  $R$  be a commutative ring,  $M$  an  $R$ -module and  $N$  a submodule of  $M$ . Then,  $N[x]$  is a primal submodule of  $M[x]$  if and only if  $N$  is a primal submodule of  $M$  and  $M/N$  is a McCoy  $R/(N :_R M)$ -module.*

**Corollary 1.6.** *Let  $R$  be a commutative ring and let  $M$  be an  $R$ -module. Then  $M[x]$  is a primal  $R[x]$ -module if and only if  $M$  is a primal  $R$ -module and  $M$  is a McCoy  $R$ -module.*

**Proposition 1.7.** *Let  $R$  be a commutative ring and let  $M$  be an  $R$ -module. If  $N$  is an irreducible submodule of  $M$ , then  $M/N$  is a McCoy  $R$ -module.*

A submodule  $N$  of  $M$  is called essential if for ever nonzero submodule  $K$  of  $M$ ,  $N \cap K \neq 0$ . A nonzero submodule  $N$  of  $M$  is called uniform if every nonzero submodule of  $M$  contained in  $N$  is essential. So  $M$  is uniform if every nonzero submodule of  $M$  is essential in  $M$ . We are going to prove the main result of the paper. First we need the following Proposition.

**Proposition 1.8.** *Let  $R$  be a commutative ring. Then every uniform  $R$ -module is primal.*

**Example 1.9.** *Assume that  $R = \mathbb{Z}$  and consider the  $\mathbb{Z}$ -module  $M = \mathbb{Z}_2 \times \mathbb{Z}_2$ . Since  $Z_R(M) = 2\mathbb{Z}$ ,  $M$  is a primal  $R$ -module. While  $M$  is not a uniform  $R$ -module. This example shows that the converse of the Proposition 1.8 is not necessarily true.*

**Theorem 1.10.** *Let  $R$  be a commutative ring. If  $M$  is a primal  $R$ -module of finite Goldie dimension, then  $M$  is McCoy.*

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