# CBS Finite Element Model for Shallow Water Problems

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## 1. Introduction

Numerical study of flow behavior in rivers and coasts has an interesting range of applications in fields such as river hydraulics, environmental hydraulics and other similar activities. In this work the formulation of a finite element numerical model for the shallow water equations is introduced and the model is tested using some standard examples cited in the literature.
The depth integrated shallow water equations govern the hydrodynamics in the shallow water bodies and one of suitable numerical techniques of these PDEs is the CBS finite element algorithm.
The foundation of the algorithm is the fractional step method initially introduced by Chorin [1] in the finite difference context for the incompressible Navier-Stokes equations. The algorithm permits some interesting and useful advantages. Firstly, it provides a critical time-step in terms of the current velocity instead of the wave celerity. It is a relevant property for low Froude number problems. Secondly, the procedure allows the application of the standard Galerkin method along the characteristics due to the split of the pressure type terms. Finally, the most important advantage of the procedure is its capability for using in the both subcritical and supercritical flows.
This method firstly introduced by Zienkiewicz [5] for modeling of shallow water equations in the finite element context. Over the past decade, many investigations have demonstrated the efficiency of this method for shallow water problems. Notable studies on this subject have been carried out by Ortiz et al. [2, 3 and 5]. This paper is devoted to the description of the CBS finite element method capabilities for modeling of some interesting problems in the shallow water field.

## 2. Mathematical model

Modeling flow hydrodynamics in shallow water bodies requires the prediction of water depth and depth averaged velocities in $x$ and $y$ directions. To do this, following shallow water equations in the depth integrated form can be written using the summation convention as

$$\frac{\partial h}{\partial t}+\frac{\partial U_i}{\partial x_i}=0 \qquad (1)$$

$$\frac{\partial U_i}{\partial t}+\frac{\partial F_{ij}}{\partial x_j}+\frac{\partial p}{\partial x_i}+Q_i=0 \qquad (2)$$

where $(i, j = 1, 2)$; $U$ and $h$, the well-known hydrodynamics variables are, respectively, $U_i = hu_i$ that $u_i$ is the $i$ component of the average velocity over the depth and $h$ the total height of water; $F_{ij} = hu_iu_j$ is the $i$ component of the $j$ flux vector and the pressure type variable $p$ is given by

$$p=\frac{1}{2}g\left(h^2-H^2\right) \qquad (3)$$

where $H$ is the depth of water measured from an arbitrary horizontal datum. The variables $h$ and $H$ are related by: $h = H + \eta$, where $\eta$ is the elevation of the free surface with respect to the arbitrary horizontal datum. Finally, $Q_i$ represents the $i$ component of a source vector given by

$$Q_i = -g\left(h-H\right)\frac{\partial H}{\partial x_i} + g\frac{u_i\left|u\right|}{C^2 h} + r_i - \tau_i \tag{4}$$

The depth integrated source terms in Equation (4) come, respectively, from bottom slope, bottom friction (Chezy's formula), Coriolis term: $r_1 = -f\,U_2$, $r_2 = f\,U_1$ where $f$ is the Coriolis factor ($f = 2\Omega sin\theta$, $\theta$: latitude of the fluid element and $\Omega = 7\times10^{-5}$ $s^{-1}$ for the earth), and wind tractions $\tau_i$.

## 3. A split method based on characteristics

The time discretization for Equations (1) and (2) is performed by proceeding along the Characteristics and can be written as follows

$$\frac{1}{c^2}\frac{\Delta p}{\Delta t} + \frac{\partial U_i^n}{\partial x_i} + \frac{\partial\left(\Delta U_i\right)}{\partial x_i} = 0 \tag{5}$$

$$\frac{\Delta U_i}{\Delta t} = -\left[\frac{\partial F_{ij}}{\partial x_j} + Q_i\right]^n + \frac{\Delta t}{2}\left[u_k\frac{\partial}{\partial x_k}\left(\frac{\partial F_{ij}}{\partial x_j} + Q_i\right)\right]^n - \frac{\partial p^{n+\theta_2}}{\partial x_i} \tag{6}$$

where

$$\frac{\partial p^{n+\theta_2}}{\partial x_i} = \left(1-\theta_2\right)\frac{\partial p^n}{\partial x_i} + \theta_2\frac{\partial p^{n+1}}{\partial x_i} - \left(1-\theta_2\right)\frac{\Delta t}{2}u_k\frac{\partial}{\partial x_k}\left(\frac{\partial p^n}{\partial x_i}\right) \tag{7}$$

($i$, $j$, $k$ =1, 2), $\Delta U_i$ and $\Delta p$ denotes the increments of the variables over a time-step $\Delta t$ and the wave celerity $c$ for long waves relates $p$ with the total height of water as

$$c^2 = \frac{dp}{dh} = gh \tag{8}$$

and ($0 \le \theta_1,\ \theta_2 \le 1$). The method is completed by the elimination of $\Delta U_i$ in the discretized continuity equation (5) by computing the divergence of Equation (6) and replacing the obtained equations into Equation (5). The following 'self adjoint' type equation for the variable $p$ is defined as

$$\frac{1}{c^2}\frac{\Delta p}{\Delta t} - \theta_1\theta_2\frac{\partial}{\partial x_i}\frac{\partial\left(\Delta p\right)}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(U_i^n + \theta_1\Delta U_i^*\right) + \theta_1\Delta t\frac{\partial}{\partial x_i}\frac{\partial p^n}{\partial x_i} \tag{9}$$

The 'intermediate' variable $\Delta U_i^*$ represents the first two terms in square brackets of Equation (6), and is obtained explicitly. The practical procedure for the computation of $p^{n+1}$ and $U_i^{n+1}$ (at the time $(n+1)\Delta t$ ) is conducted by the following steps: ($i$) computation of the intermediate variable, ($ii$) computation of $\Delta p$ and ($iii$) correction of the momentum components by means of the complete momentum equation (6).

## 4. Model test

In order to illustrate the efficiency of the described model, the following well-known examples are reported. For the sake of simplicity, these examples are all pseudo-one-dimensional and hence triangular uniform meshes are used along the entire of the solution domains. Since there are not the exact solutions for these situations, the results are evaluated qualitatively.

The first example, illustrated in Fig. 1, shows the progress of a solitary wave onto a frictionless shelving beach. Fig. 2 shows the obtained wave profiles during its propagation along the beach. The results indicate well the progressive steeping of the wave often obscured by other schemes that are very dissipative.

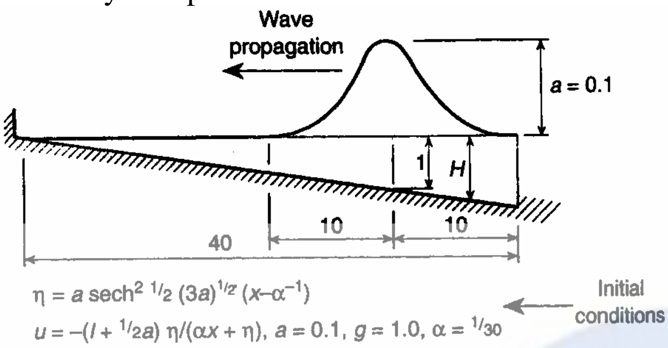


Fig. 1- Problem statement and its initial conditions

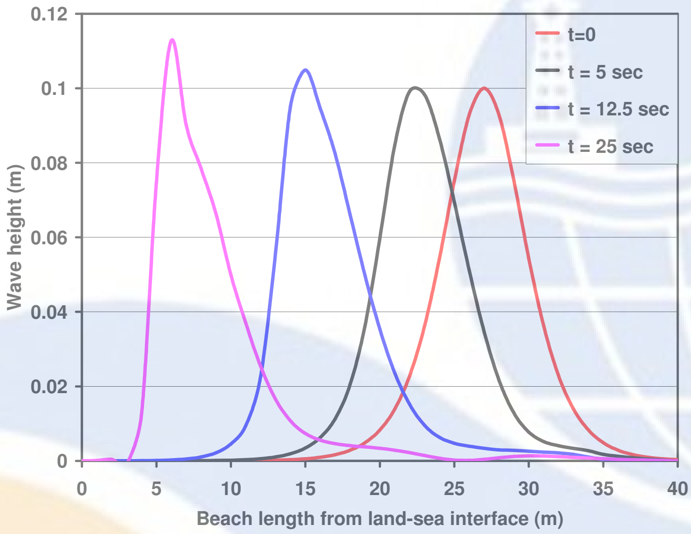


Fig. 2- Wave profiles along the beach

In the second problem, a dam is considered at the middle of a rectangular flat channel and the flow parameters after removing the dam are computed. The obtained results employing 40 equal elements along the channel are shown in Fig. 3 and 4 compare favorably with published results obtained using other numerical techniques.

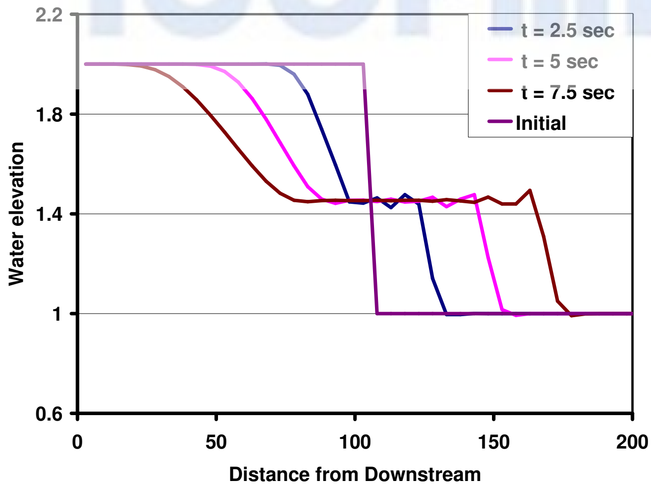


Fig. 3- Water surface profiles after removing the dam

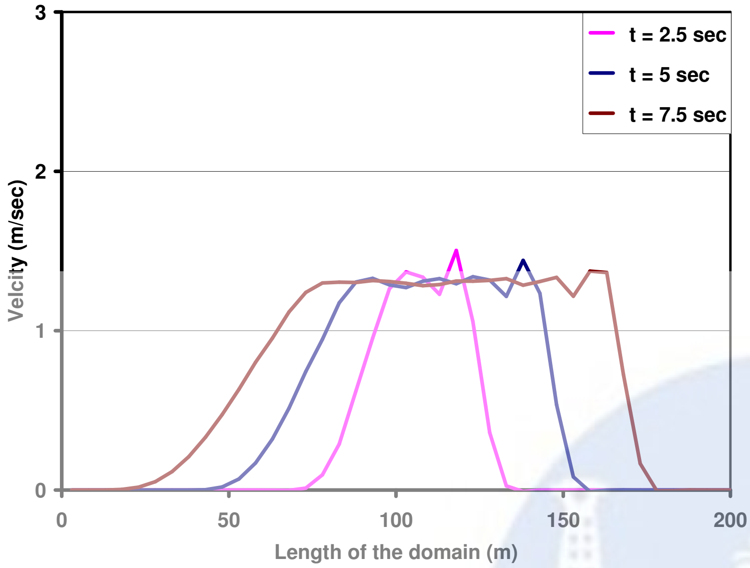


Fig. 4- Velocity profiles along the entire of the domain

The problem of bore propagation along the rectangular flat channel is the last example studied here. The depth of 1 *m* and velocity of 1 *m/sec* are given as the initial conditions. Boundary conditions are imposed as the flow velocity of 1 *m/sec* in the upstream and a sinusoidal rising water elevation with period of 30 *sec* in the downstream boundary of the channel. After $t = 30$ *sec* the water elevation is imposed as 2 *m* at this boundary. Obtained results for water surface profiles and flow velocity during the bore propagation are shown in Fig. 5 and 6. The results are in good agreement with the reported results in the literature such as [6].

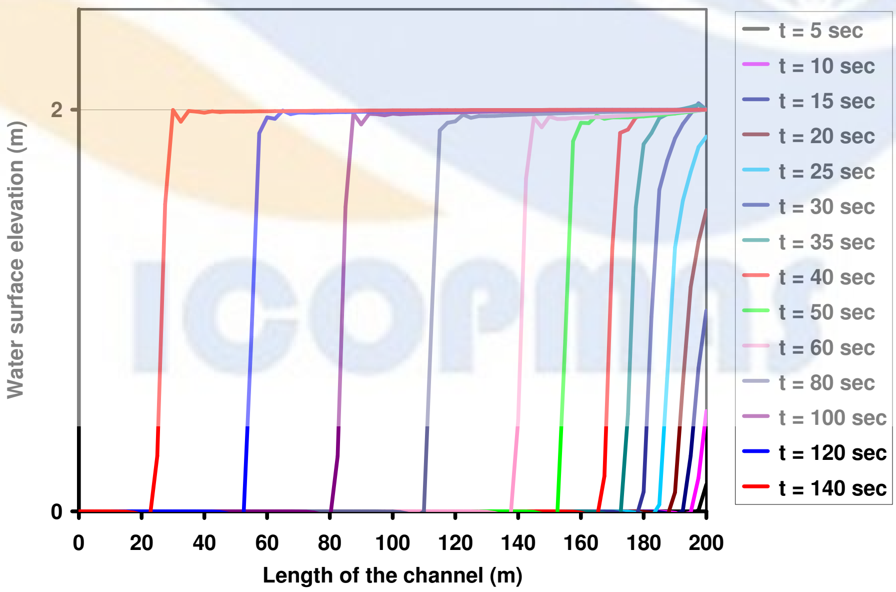


Fig. 5- Water profiles along the channel due to bore propagation

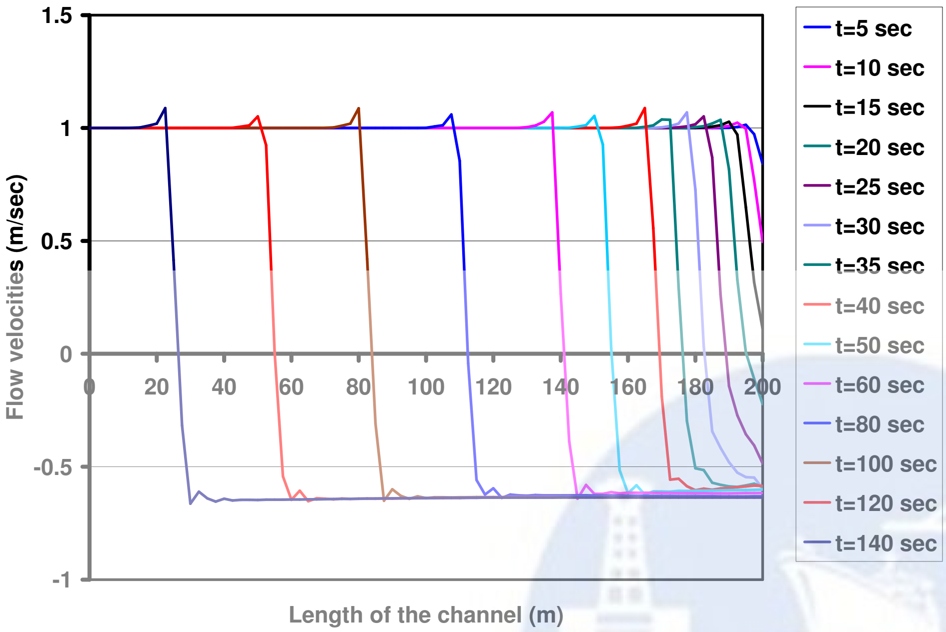


Fig. 6- Flow velocities along the channel due to bore propagation

## 5. Conclusion

The CBS finite element model was used for the numerical solution of shallow water equations presented has some remarkable features. The conjunction of a stability limit dependent on the flow velocity instead of on the wave celerity with an implicit computation of diffusion source terms to keep the convective limiting time-step is especially useful for modeling of real long term problems. The developed model was evaluated by some test problems and produced satisfactory results. In spite of the highly non-linear nature of the flow in these examples, the computational results indicate the favorable performance of the used procedure for modeling of the shallow water problems. As an important result, the described model can safely be used as an efficient tool for real applications in hydraulic engineering involving complex flow situations.

## 6. References


[1] Chorin, A., “Numerical solution of the Navier Stokes equations.”, *Math. Comput.*, Vol. 22, pp. 745–762, 1968.

[2] Ortiz, P., Zienkiewicz, O.C. and Szmelter, J., “Hydrodynamics and transport in estuaries and rivers by the CBS finite element method.”, *International Journal for Numerical Methods in Engineering*, Vol. 66, pp. 1569-1586, 2006.

[3] Ortiz, P., Zienkiewicz, O.C. and Szmelter, J., “CBS finite element modeling of shallow water and transport problems.”, *European Congress on Computational Methods in Applied Sciences and Engineering*, pp. 1–14, 2004.

[4] Zienkiewicz, O. C., Nithiarasu, P. R. Codina, M. Vazquez and Ortiz, P., “An efficient and accurate algorithm for fluid mechanics problems: The characteristic based split procedure.”, *International Journal of Numerical Methods in Fluids*, Vol. 31, pp. 359–392, 1999.

[5] Zienkiewicz, O. C., Ortiz P. “A split characteristic based finite element model for the Shallow Water equations.”, *International Journal for Numerical Methods in Fluids, Vol.* 20, pp. 1061–1080, 1995.

[6] Zienkiewicz, O. C. and Taylor, R. L. “The finite element method: Volume 3: Fluid dynamics.”, *Butterworth Heinemann*, pp. 218–228, *2000*.