Wave-Current Interaction and its Influence on Sediment Transport*

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۱. Abstract

The interactions between waves and currents were evaluated and their potential influence on sediment transport assessed. To consider wave-current interactions, wave and current fields were individually examine and then combined. Flow fields were explored using an available hydrodynamic model, the Environmental Fluid Dynamics Code (Hamrick, ۱۹۹۶; Jin et al. ۲۰۰۰). Three flow fields were considered: a strong current (a river), a moderate current (a bay or harbor), and a weak current (small lake with wind-driven circulation). Wave fields were explored using small amplitude (linear) wave theory. The combined effect of wave-current interactions was explored using the approach of Grant and Madsen (۱۹۷۹). These analyses indicate that, in the presence of currents, waves dramatically enhance sediment transport. Even when a current is weak, the combined effect of waves and currents can resuspend and transport sediments. For the conditions explored, bottom shear stresses from wave-current interaction were greater than the shear stresses from currents alone by an order of magnitude or more.

۲. Introduction

The sediments of many river, lakes, and estuaries are contaminated with chemicals that pose risks to human health and the environment. Successful management of environmental risks in coastal zones affected by contaminated sediments is often a complex undertaking since interactions between water and sediment can transport particles and associated chemicals great distances. To select the most beneficial and cost-effective means to address risks posed by contaminated sediments, it is necessary to determine where particles will be transported over time. It is therefore necessary to understand the factors that influence sediment transport in coastal zones.

During the research, the major topics examined were wave generation by astronomical and meteorological forces and wave transformation. As waves approach the shoreline, the bottom velocities originating from the oscillatory motion of progressive waves exert shear forces at the sediment-water interface. These shear forces have the potential to erode sediment. Despite the high

* Some parts of the Article has been omitted to make it more shortened matching as a conference paper (such as motivation and objective sections and some parts from theoretical sections), these parts are available on request.

potential for erosion, progressive waves do not efficiently transport sediment since there is little or no net motion over a wave period. However, in the coastal environment, currents also occur. Currents have well-defined unidirectional motion so even small currents can transport sediments away from an erosion site. Nonetheless, unlike waves, the shear forces exerted by currents can be much smaller than those exerted by waves. Therefore, despite the ability to transport sediment, currents may have limited ability to erode particles from the bed due to the small magnitude of applied shear forces. Consequently, where currents are weak, the potential for strictly current-driven sediment erosion and transport is small. However, when waves and currents interact, their effects combine and the potential for erosion and transport is greatly increased. Waves erode sediments from the bed and currents transport suspended particles away from the erosion site.

۳. Methods

To consider wave-current interactions, the wave and flow fields of an area must be described. Once these fields are defined, the interactions between waves and currents can be evaluated and their influence on sediment transport assessed. Flow fields were explored using an available hydrodynamic model, the Environmental Fluid Dynamics Code (Hamrick, ۱۹۹۶; Jin et al. ۲۰۰۰). Three representative flow fields were considered. Wave fields were then explored using small amplitude wave theory. A brief literature review indicated that Grant and Madsen (۱۹۷۹) conducted formative research into wave-current interactions. That research is referenced in many of the journal articles on this topic. Wave-current interactions were explored using the approach of Grant and Madsen (۱۹۷۹).

۳.۱. Current Fields (Hydrodynamic Simulations)

Numerical simulations of water flow were performed using the Environmental Fluid Dynamics Code (EFDC) (Hamrick, ۱۹۹۶). As described by Jin et al. (۲۰۰۰), EFDC solves the three-dimensional, unsteady, free surface, turbulence averaged equations of motion for a variable density fluid. Dynamically coupled transport equations for turbulent kinetic energy and turbulent length scale are solved, as is a temperature transport equation. Fluid density is expressed as a function of pressure and temperature by a state equation to couple the motion and temperature transport equations. A stretched, or sigma, vertical coordinate and Cartesian horizontal coordinates are used. The governing equations are (Jin et al., ۲۰۰۰):

$$\frac{\partial H}{\partial t}+\frac{\partial (Hu)}{\partial x}+\frac{\partial (Hv)}{\partial y}+\frac{\partial \omega}{\partial \sigma}=Q_H \qquad (۱)$$

$$\frac{\partial(Hu)}{\partial t}+\frac{\partial(Huu)}{\partial x}+\frac{\partial(Huv)}{\partial y}+\frac{\partial(u\omega)}{\partial\sigma}-fHv=-H\frac{\partial(p+p_{atm}+gz)}{\partial z}+\left(\frac{\partial z_b}{\partial x}+\sigma\frac{\partial H}{\partial x}\right)\frac{\partial p}{\partial\sigma}+\frac{\partial}{\partial\sigma}\left(\frac{A_v}{H}\frac{\partial u}{\partial\sigma}\right), \quad (۲)$$

$$\frac{\partial(Hv)}{\partial t}+\frac{\partial(Huv)}{\partial x}+\frac{\partial(Hvv)}{\partial y}+\frac{\partial(v\omega)}{\partial\sigma}-fHu=-H\frac{\partial(p+p_{atm}+gz)}{\partial y}+\left(\frac{\partial z_b}{\partial y}+\sigma\frac{\partial H}{\partial y}\right)\frac{\partial p}{\partial\sigma}+\frac{\partial}{\partial\sigma}\left(\frac{A_v}{H}\frac{\partial v}{\partial\sigma}\right), \quad (۳)$$

$$\frac{\partial p}{\partial\sigma}=-gH\frac{(\rho_w-\rho.)}{\rho.} \quad (۴)$$

where: x, y, z = horizontal (x, y) and vertical (stretched, sigma) coordinates; H = total water column depth; t = time; u, v, omega = velocity components in the x, y, and sigma directions; Q_H = water inflow/outflow (source/sink); f = Coriolis acceleration; z_b = bottom elevation; p, p_{atm} = excess hydrostatic pressure, atmospheric pressure; ρ_w, $\rho.$ = water density, reference water density; A_H = momentum diffusion coefficient; and g = gravitational acceleration constant.

Equation (۱) represents the conservation of mass (continuity). Equations (۲)-(۳) represent the conservation of momentum in the x, y, and sigma directions. The sigma coordinate is a surrogate for the z (vertical) direction. The last term on the right hand side of Equations (۲)-(۳) represents the variation of shear stress (τ) with depth. Equation (۴) represents pressure variation with depth. For simplicity, two-dimensional (vertically averaged) test simulations explored. For two-dimensional cases, the governing equations are simplified as all values in the sigma direction assume average values. The momentum equations can be further simplified, as the Coriolis force is typically negligible in small water-bodies.

To develop a basic understanding of EFDC operation and generate typical flow fields, three test simulations were constructed: ۱) a narrow channel with an obstruction; ۲) a small, open basin; and ۳) a small, closed basin. These simple hydrodynamic simulations were intended to be representative of environmentally relevant conditions. The narrow channel with an obstruction could represent a typical river with an island. Note that the obstruction could also be set up to represent a bridge piling, pier, or other structure. The small, closed basin could represent and inland lake with wind-driven circulation patterns. The small, open basin could represent a harbor or bay with flow and waves that enter from a connecting water-body.

۳.۲. Wave Field

In addition to currents, assessment of wave-current interactions requires specification of wave fields. For simplicity, wave motion may be represented by small amplitude wave theory (Grant and

Madsen, ۱۹۷۹). Linear wave theory was discussed extensively in many of literatures. The equations describing small amplitude wave kinematics and wave dispersion are (Kamphuis, ۲۰۰۰):

$$\eta = \frac{H}{۲}\cos(kx - at) \qquad (۵)$$

$$u = \frac{\pi H}{T}\frac{\cosh[k(z+d)]}{\sinh(kd)}\cos(kx - at) \qquad (۶)$$

$$w = \frac{\pi H}{T}\frac{\sinh[k(z+d)]}{\sinh(kd)}\sin(kx - at) \qquad (۷)$$

$$A = \frac{H}{۲}\frac{\cosh[k(z+d)]}{\sinh(kd)} \qquad (۸)$$

$$B = \frac{H}{۲}\frac{\sinh[k(z+d)]}{\sinh(kd)} \qquad (۹)$$

$$u_b = \frac{۵}{۴}\frac{a^{۲} k\omega}{\sinh^{۲}(kd)} \qquad (۱۰)$$

$$\omega^{۲} = kg\tanh(kd) \qquad (۱۱)$$

where: η = water surface fluctuation (position) relative to the mean water level; H = wave height; k = wave number; ω = angular wave frequency; L = wave length; T = wave period; u, w = horizontal and vertical velocity components; d = water depth; z = vertical location relative to the mean water level; a = wave amplitude = H/۲; A, B = horizontal and vertical orbital excursion distances; u_b = near-bottom velocity (mass transport); and g = gravitational acceleration constant.

Linear wave theory assumes that the dimensionless wave steepness (H/L) and relative wave height (H/d) are much less than one (<<۱), the pressure along the air-water interface is constant, and the flow is frictionless, irrotational, and incompressible. For further simplicity, a uniform wave field was assumed for deep-water conditions of H = ۰,۳۰ meters, and T = ۸ seconds. Note that a wave field could alternatively have been generated using the SMB or JONSWAP methods for specified wind and fetch conditions as presented by Kamphuis (۲۰۰۰).

۳.۳. Wave-Current Interactions

Wave-current interactions were explored using the approach of Grant and Madsen (۱۹۷۹). A brief literature review indicated that Grant and Madsen (۱۹۷۹) conducted formative research into wave-current interactions. Their research is cited as the basis for much of the subsequent research into wave-current interactions. The following description of wave-current interaction was summarized from Grant and Madsen (۱۹۷۹).

The interactions between waves and currents are nonlinear. The combined effects of currents and waves cannot be computed by simple superposition of the individual effect of waves and currents alone. In natural systems, different time scales are associated with currents and waves. Relative to waves, currents can be considered slowly varying, essentially steady flows. In contrast, the oscillatory motions of waves generate unsteady flows. Near the sediment bed, the thickness of the current boundary layer flow is much greater than the boundary layer flow generated by unsteady wave motion. When the combined effects of waves and currents are considered, the flow in the wave boundary layer experiences resistance from the sediment bed as well as the surrounding current flow in which the wave occurs. As summarized from Grant and Madsen (۱۹۷۹), the basic equations describing wave-current interactions are:

$$\left|\tau_{b,\max}\right| = \frac{۱}{۲} f_{cw}\rho\left[۱+\left(\frac{|u_c|}{|u_w|}\right)^{۲}+۲\left(\frac{|u_c|}{|u_w|}\right)\cos\phi_c\right]|u_w|^{۲} = \rho\left|u^{*}_{cw}\right|^{۲} \qquad (۱۲)$$

$$\frac{k_{bc}}{k_b} = \left[۲۴+\frac{|u^{*}_{cw}|}{|\omega . k_b|}\right]^{\beta}, \qquad \beta = \left[۱-\frac{\ln(۳۰\delta_w/k_{bc})}{\ln(۳۰\delta_w/k_b)}\right] \qquad (۱۳)$$

where: $\tau_{b,\max}$ = maximum bottom shear stress of the combined wave and current flow; f_{cw} = friction factor associated with the combined wave and current flow; ρ = water density; ϕ_c = angle between the directions of current and wave fields relative to the direction of wave propagation, u_w = maximum near-bottom orbital wave velocity (= u_b); u_c = current velocity (at a reference height above the sediment bed); u_{*cw} = shear (friction) velocity of the combined wave and current flow; k_b = physical bed roughness; k_{bc} = apparent bed roughness; ω = wave frequency; and δ_w = thickness of the wave boundary layer.

Equation (۱۲) represents a quadratic drag law. The shear stress in this equation is based on the maximum wave orbital velocity to represent conditions in the wave boundary layer as it occurs within the current boundary layer. In this formulation, the combined effect of waves and currents is expressed through the friction factor, f_{cw}. The value of f_{cw} continually changes as a function of the continuous modification of the bed roughness by the flow. Through the shear velocity of the combined flow, f_{cw} is related to the bed roughness. The physical roughness of a sediment bed is related to flow resistance (drag) caused by the size of the individual grains that comprise the bed (skin friction). The apparent bed roughness is an additional drag component caused by pressure gradients attributable to the configuration of the bed (form drag). The action of waves in a current field causes the flow in the wave boundary to experience additional drag (in the form of mechanical energy dissipation in the form of turbulent eddies) that can be expressed as an increase in the apparent bed roughness. The apparent bed roughness can be many times larger than the physical roughness of the sediment bed. Determination of f_{cw} requires iterative solution of these nonlinear equations. The solution to these equations for f_{cw} is developed by Grant and Madsen (۱۹۷۹).

۴. Results

۴.۱. Current Fields (Hydrodynamic Simulations)

Hydrodynamic models are frequently used to provide detailed representations for flows (Lane et al. ۱۹۹۹). The three hydrodynamic simulations explored represent a range of environmentally relevant conditions:

Simulation ۱ represents a situation common in riverine environments. When considered in two dimensions, obstructions such as bridge pilings, piers, and islands cause the flow field to separate at the upstream edge of the obstruction and converge downstream. Along either side of the obstruction, water velocities, shear stresses, and the corresponding capacity to erode sediments from the bed and transport those particles downstream increases. When further considered in three dimensions, flow past an obstruction can give rise to flow structures such as horseshoe vortices that can greatly increase sediment bed scour (Ahmed and Rajaratnam, ۱۹۹۸). With respect to wave-current interactions, rivers tend to be current dominated systems. Limited fetch lengths typically prevent significant wave generation. In such situations, waves are generally expected to contribute little to sediment transport. However, at locations where waters pool (such as behind large dams) or near the river mouth, current strength can decrease and the potential for wave generation increase. At such locations waves may nonetheless contribute to sediment transport. Typical current velocities from this simulation were roughly ۵۰ cm/s. This represents a relatively strong current.

Simulation ۲ represents a situation common in harbors and bays connected to larger water-bodies. Because of the open boundary representing a connection to a larger water-body, the magnitudes of currents in this simulation are expected to fall between the river (strong current) and wind-driven lake (weak current). However, waves are nonetheless expected to be the dominant component of sediment transport near the shoreline. Typical current velocities from this simulation were roughly ۵ cm/s. This represents a moderate current.

Simulation ۳ represents a situation common in small inland lakes where currents are generated by wind action. Winds moving over the lake surface can generate waves and currents. The wind-generated currents are expected to be small due to limited fetch lengths. Under these conditions, waves are expected to be the dominant component of sediment transport, especially close to the shoreline. However, current magnitudes were nonetheless surprising small. Typical current velocities from this simulation were less than ۰٫۱ cm/s. This represents a very weak current.

۴.۲. Wave Field

The wave field was generated from the parameters in Table ۱ using linear wave theory. The waves described by these parameters are representative of gravity waves with periods between ۱ and ۳۰ seconds that occur in most water-bodies. The wave height is representative of waves generated by a moderate to fresh breeze as measured by the Beaufort scale assuming fetch limited conditions. Now, if we compute the bottom velocities from eq. (۱۰) (linear wave theory), it can be seen that they increase significantly as a wave moves from deep water and shoals. However, as waves move from deep to shallow water and shoal, linear wave theory breaks down. For example, Madsen and Mei (۱۹۶۹) showed that a single solitary wave moving up a slope onto a shelf will break into two additional solitary waves with the third wave disintegrating into a train of weak linear waves.

۴.۳. Wave-Current Interactions

The combined wave and current fields were used to assess the influence of wave-current interactions on sediment transport. One major factor that controls the magnitude of sediment transport is the bottom shear stress (τ_b). Currents may be considered steady (constant over time) flows. For currents alone, a quadratic drag law can be used to represent τ_b:

$$|\tau_b| = \frac{۱}{۲} f_c \rho |u_c|^{۲} = \rho |u_c^*|^{۲} \quad (۱۵)$$

where: τ_b = bottom shear stress of the current; f_c = friction factor associated with the current; ρ = water density; u_c = current velocity (at a reference height above the sediment bed); u_{*c} = shear (friction) velocity of the current.

The factor ۰,۵*ρ*f_c represents a drag coefficient (C_d). For a cohesive sediment bed, a typical value for C_d is approximately ۰,۰۰۲۵, where ρ has units of g/cm^۳ and u_c units of cm/s. From the hydrodynamic simulations, typical mean current velocities ranged from ۵۰ cm/s (river) to ۵ cm/s (open basin) to ۰,۱ cm/s (closed basin). Corresponding shear stresses for these velocities range from ۶,۲۵ dynes/cm^۲ (river) to ۰,۰۶۲ dynes/cm^۲ (open basin) to ۲,۵ x ۱۰$^{-۷}$ dynes/cm^۲ (closed basin). For comparative purposes, critical shear stresses needed to resuspend sediments range from ۰,۸ dynes/cm^۲ for a ۱۰۰ um sand particle to ۱ dyne/cm^۲ for typical cohesive sediment. Generalizing from these results, river currents have the power to resuspend and transport sediments (such as during flood conditions) whereas the weaker currents found in lakes have less power to do that.

Wave motion is oscillatory and resultant flows are unsteady (vary over time). For waves alone, shear stress resulting from the (instantaneous) maximum wave orbital velocity may also be represented by Equation (۱۵). Maximum wave orbital velocities at the sediment water interface vary with water depth (d) as waves shoal. For the wave field described in previous sections, typical maximum orbital velocities at the sediment-water interface ranged from ۱۱ cm/s (d = ۱۰ m) to ۱۹ cm/s (d = ۵ m) to ۴۵ cm/s (d = ۱ m). Corresponding shear stresses for these velocities range from ۰,۳۰ dynes/cm^{۲} (۱۰ m) to ۰,۹۲ dynes/cm^{۲} (۵ m) to ۵,۰۶ dynes/cm^{۲} (۱ m). Again, for comparative purposes, critical shear stresses for resuspension are roughly ۰,۸ to ۱ dyne/cm^{۲}. Generalizing from these results, even moderate waves (H = ۰,۳۰ m) have the power to resuspend sediments in shallow (d < ۵ m) water. However, because wave motion is oscillatory (i.e. no net motion over a wave period), waves alone have less ability to transport sediments than do currents.

The combined effect of waves and currents may be computed from Equations (۱۲)-(۱۳). For simplicity, the solution for f_{cw} presented by Grant and Madsen (۱۹۷۹). Assuming a co-directional flow (i.e. currents and wave travel in the same direction so ϕ_c = ۰ degrees), f_{cw} values ranged from ۰,۰۳ to ۰,۱۲ for the combination of current and maximum wave orbital velocities described above. Note that the apparent bed roughness values were also assumed to further simplify this assessment. For more complete results, the apparent bed roughness should be dynamically computed.

Table ۱. Combined wave-current interaction assessment results.

Current Type	*Water Depth m*	u_c *cm/s*	u_w *cm/s*	u_c / u_w	f_{cw}	$\tau_{b,max}$ *dynes/cm*^{۲}	τ_b *dynes/cm*^{۲}	$\tau_{b,max} / \tau_b$
Channel (River)	۱۰	۵۰	۱۱	۴,۵۴۵	۰,۰۳۰	۵۵,۸	۶,۳	۸,۹۳
	۵	۵۰	۱۹	۲,۶۰۴	۰,۰۴۰	۹۵,۸		۱۵,۳
	۱	۵۰	۴۵	۱,۱۱۱	۰,۰۸۰	۳۶۱۰		۵۷۸
Open Basin (Bay)	۱۰	۵	۱۱	۰,۴۵۴	۰,۰۵۰	۶,۴۰	۰,۰۶	۱۰۲
	۵	۵	۱۹	۰,۲۶۰	۰,۰۷۰	۲۰,۵		۳۲۸
	۱	۵	۴۵	۰,۱۱۱	۰,۱۲۰	۱۵۰		۲۴۰۰
Closed Basin (Small Lake)	۱۰	۰,۱	۱۱	۰,۰۰۹	۰,۰۷۰	۴,۳۱	۲,۵ x ۱۰$^{-۷}$	~۱۰^{۷}
	۵	۰,۱	۱۹	۰,۰۰۵	۰,۰۹۰	۱۶,۸		~۱۰^{۷}
	۱	۰,۱	۴۵	۰,۰۰۲	۰,۱۲۰	۱۲۲		~۱۰^{۸}

Combined wave-current bottom shear stresses are presented in Table ۱. The combined occurrence of waves and currents can increase maximum bottom shear stresses by an order of magnitude or more. It is worth noting that significant wave generation in a narrow channel is unlikely as a result of fetch limitation. The results presented for the river case may be considered largely hypothetical. Nonetheless, in areas were a river joins a larger water body (i.e. the river mouth), it is possible for waves generated in the connected water-body to move into the channel. However, under such conditions the wave and current field may be in opposition (i.e. ϕ_c = ۱۸۰ degrees). Based on the theory developed by Grant and Madsen (۱۹۷۹), ϕ_c is not defined for angles greater than ۹۰ degrees so it is unclear how to compute interactions for opposing wave and current flows.

۵. Discussion and Conclusions

In river, currents are expected to be strong and the predominant factor that controls sediment erosion and transport. In the nearshore areas of bays and lakes, currents are generally weaker and waves are expected to significantly contribute to sediment erosion and transport. The results of this assessment indicate that waves may increase bottom shear stresses by an order of magnitude.

It is worth noting that linear wave theory was used for this analysis. Linear wave theory breaks down as waves move from deep to shallow water and shoal. The wave shoaling process is itself nonlinear. The formulations presented for the wave orbital velocity and bottom shear stress may not accurately represent conoidal or solitary waves, especially in the presence of currents.

The dynamics of wave and current motions are altered in situations where the sediment bed is mobile (erodible) and flows generate bed forms (surface undulations such as ripples) (Kobayashi and Madsen, ۱۹۸۵a,b). Bed forms increase bed roughness and change the velocity profile of the flow in the vicinity of the bed, which in turn causes further modification of the bed. In natural systems, flows are often unsteady and non-uniform. As a result, bed modification by flows is continuous. McLean et al (۱۹۹۹) further indicate that more detailed consideration must be given to estimates of the skin friction component of the total drag in order to accurately represent bed mobility and its effect on wave-current interactions and sediment transport.

Finally, wind-wave interactions can influence water levels (and sediment transport) across large regions. A recent application by Jones and Davies (۲۰۰۱) found that it was necessary to account for wave-current interactions in order to more accurately estimate ocean storm surges. Because storm surges are generated by wind stresses at the water surface, these researchers also found that the results they obtained where sensitive to the drag coefficient for the air-water interface.

From these simplified and conceptual results, the conclusions of this brief research are:

- In the presence of currents, waves can dramatically enhance sediment transport. For the conditions explored in this research, bottom shear stresses from wave-current interaction were an order of magnitude or greater than the shear stresses from current alone.
- Linear wave theory was used for this analysis. Linear wave theory breaks down as waves move from deep to shallow water and shoal and may not accurately represent conoidal or solitary waves, especially in the presence of currents.
- The dynamics of wave and current motions are altered when the sediment bed is erodible and bed forms are generated by the flow. Bed forms increase bed roughness and change flow velocity profiles in the vicinity of the bed, which in turn lead to further modification of the bed by the flow. More detailed consideration should be given to estimates of the skin friction component of the total drag in order to accurately represent bed mobility and its effect on wave-current interactions and sediment transport.
- Even if current are weak, the combined effects of waves and currents can resuspend and transport sediments. Nearshore, waves may be the predominant force for sediment transport.

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