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Equivalent buckling length factor of semi-rigid frames with tapered columns

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Abstract

In this paper, a new exact formulation for calculating the elastic buckling load of semi-rigid steel frames with tapered columns will be obtained. The presented methodology is based on the precise solution of the governing differential equations for buckling of the uniform and non-uniform frames. Then, the effects of the non-prismatic members, with a parabolic stiffness variation, flexibility of connections, and lateral bracing on the buckling length factor and corresponding critical load of a portal frame will be investigated parametrically. Results show the combined effect of aforenamed parameter on the critical buckling load and corresponding equivalent buckling length factor of portal steel frames is very significant.

Keywords: Buckling analysis, Tapered columns, Semi-rigid connections, Steel frames, Buckling length factor, Critical load.

1. Introduction

Tapered comparison members are extensively used in structural, mechanical and aeronautical engineering. The use of tapered members was first proposed by Amirikian [1] in 1952 for reasons of economy according to Lee and Morrell [2]. Nowadays, most of the civil engineering structures consisting of tapered columns with various cross sections to minimize the total weight and subsequently the cost of structures and sometimes to satisfy architectural and functional requirements. This paper deals with the stability analysis of tapered columns and 2-D frames consisting of linearly tapered elements with a second-order polynomial stiffness variation.

In the search for analytical solutions of the buckling of frames, many research works are based on assumed stiffness distributions. Ermopoulos and Kouanadis [3] dealt with simple portal braced and un-braced frames comprising tapered lattice columns with a second-order polynomial stiffness variation and the buckling load were established in closed-form by means of bifurcation analysis. Using the same model, Ermopoulos studied the non-linear buckling analysis of simple frames in non-sway mode [4]. Also, he presented results for tapered bars under stepped axial loads and utilized the same assumption to obtain the equivalent buckling length of non-uniform members on the basis of the slope-deflection method [5-9]. Moreover, this investigator and Raftoyiannis studied the effect of initial imperfections on the stability of tapered members [10].

Based on the previous review, it can be seen that comprehensive studied on the stability of non-prismatic column rather than the frames involved non-uniform member and no attempt has been made for considering the joint of flexibility and elastic bracing system in steel portal frames with non-prismatic members. The purpose of this study is to determine exact expression accounting for aforenamed effects with various cross sections for the critical buckling load of frame. The methodology is based on the exact solution of the governing differential equations for buckling of the uniform and non-uniform frames.

2. Buckling analysis

It is intended to analysis the portal frames shown in Fig. 1. The frame in Fig. 1(a) has two pinned supports; while the frame in Fig. 1(b) is based on two fixed supports. Columns have both length l_c, and moment of inertia is assumed to vary in the following form:

$$I_i(x_i) = I_c\left(\frac{x_i}{a}\right)^n \qquad (i = 1,3) \tag{1}$$

In this function $I_i(x_i)$ is the moment of inertia of the cross-section at a distance x_i from the origin, as shown in Fig. 1, and I_c is the moment of inertia at a distance a from the origin. According to Table (1), the shape factor, n, is equal to 2 for tapered members with varying depth and constant cross-sectional area, such as tower and open-web sections. It should be noted, for the uniform member, the shape factor n is equal to zero. The beam has length l_b, and moment of inertia I_b. Each frame is subjected to two vertical concentrated loads, P_1 and P_3, on the centerline of columns. The beam is connected to columns via semi-rigid connections. It is assumed that both beam-to-column connections, has rotational stiffness K_c. The lateral elastic support is modeled by a horizontal spring with axial stiffness K_b, which is located at the top of the right column.

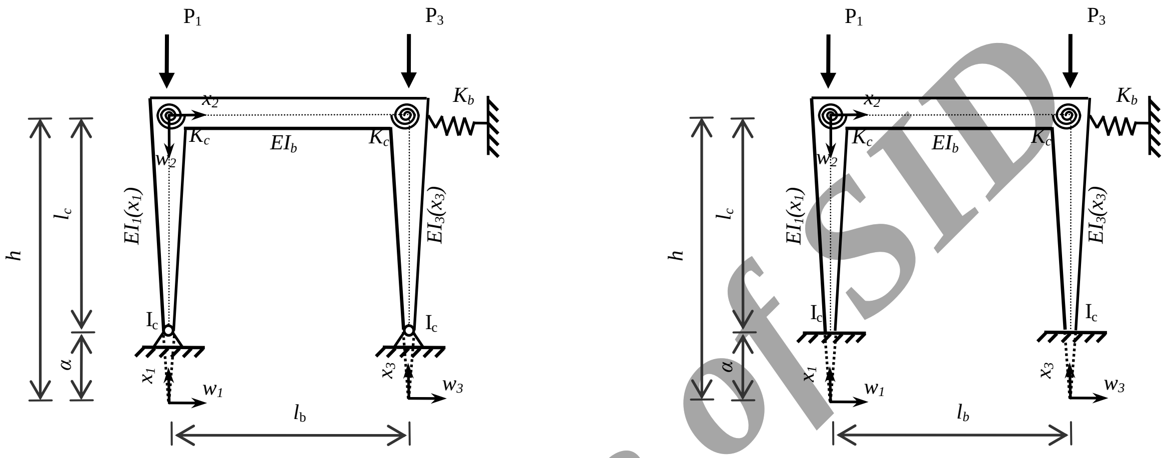


Figure 1- Geometry and sign convention of non-uniform frames with: (a) pinned supports, and (b) fixed supports.

Within the limitations of the beam-column theory, the governing forth-order differential equations for the columns and the beam are given below:

$$\left.\begin{aligned}
&\frac{d^2}{dx_1^2}\left[EI_c\left(\frac{x_1}{a}\right)^n\frac{d^2w_1}{dx_1^2}\right] + P_1\frac{d^2w_1}{dx_1^2} = 0\\
&\frac{d^2}{dx_2^2}\left[EI_b\frac{d^2w_2}{dx_2^2}\right] = 0\\
&\frac{d^2}{dx_3^2}\left[EI_c\left(\frac{x_3}{a}\right)^n\frac{d^2w_3}{dx_3^2}\right] + P_3\frac{d^2w_3}{dx_3^2} = 0
\end{aligned}\right\} \tag{2}$$

The general solutions of Eq. (2) for n = 0 and 2 are presented in Eqs. (3) and (4), respectively.

$$\left.\begin{aligned}
&w_1 = A_1\sin\rho x + B_1\cos\rho x + C_1x + D_1\\
&w_2 = A_2x^3 + B_2x^2 + C_2x + D_2\\
&w_3 = A_3\sin\rho x + B_3\cos\rho x + C_3x + D_3
\end{aligned}\right\} \quad , \quad \left(\rho = \sqrt{\frac{Pl_c^2}{EI_c}}\right) \tag{3}$$

$$\left.\begin{aligned}
w_1 &= \sqrt{\frac{x}{a}}\left\{A_1\sin\left[\rho Ln\left(\frac{x}{a}\right)\right]+B_1\cos\left[\rho Ln\left(\frac{x}{a}\right)\right]\right\}+C_1x+D_1\\
w_2 &= A_2x^3+B_2x^2+C_2x+D_2\\
w_3 &= \sqrt{\frac{x}{a}}\left\{A_3\sin\left[\rho Ln\left(\frac{x}{a}\right)\right]+B_3\cos\left[\rho Ln\left(\frac{x}{a}\right)\right]\right\}+C_3x+D_3
\end{aligned}\right\} \quad , \quad \left(\rho=\sqrt{\frac{Pa^2}{EI_c}-\frac{1}{4}}\right) \tag{4}$$

where A_i, B_i, C_i, and D_i (i=1,2,3) are integration constants to be determined using boundary and kinematic conditions.

In practice, both columns in a simple frame have the same sectional properties (i.e. $I(x_1)=I(x_3)$), and mostly loaded by equal compression forces (i.e. $P_1 \approx P_3$). Accordingly, it is assumed that $P_1=P_3=P$ and $I_1=I_3$. At this stage, by employing the boundary and kinematic conditions, and also the coming non-dimensional parameters, $\rho^2=\frac{Pa^2}{EI_c}-\frac{1}{4}$, $\nu=\frac{I_c l_b}{I_b h}$, $r=\frac{a}{h}$, $K_b^*=\frac{K_b h^3}{EI_c}r^2$, $K_c^*=\frac{K_c l_b}{EI_b}$, $S=\sqrt{\frac{1}{r}}\sin\left(\rho\ln(\frac{1}{r})\right)$, $C=\sqrt{\frac{1}{r}}\cos\left(\rho\ln(\frac{1}{r})\right)$, the following system of dimensionless equations can be found, when the shape factor, n, is equal to 2:

$$\left.\begin{aligned}
&B_1+r\overline{C}_1+D_1=0\\
&-(\rho^2+1/4)B_1=0\\
&B_3+r\overline{C}_3+D_3=0\\
&-(\rho^2+1/4)B_3=0\\
&\nu(\overline{A}_2\nu+\overline{B}_2)+\overline{C}_2=0\\
&S(A_1-A_3)+C(B_1-B_3)+(\overline{C}_1-\overline{C}_3)+(\overline{D}_1-\overline{D}_3)=0\\
&K_b^*(A_3S+B_3C+\overline{C}_3+D_3)-(\overline{C}_1+\overline{C}_3)(\rho^2+1/4)=0\\
&(1/r)(\rho^2+1/4)(A_1S+B_1C)+2\overline{B}_2r=0\\
&K_c^*\left[C(A_1\rho+B_1/2)+S(A_1/2-B_1\rho)+(\overline{C}_1-\overline{C}_2)\right]+2\overline{B}_2\nu=0\\
&-2r(3\overline{A}_2\nu+\overline{B}_2)+(1/r)(\rho^2+1/4)(A_3S+B_3C)=0\\
&2\nu(3\overline{A}_2\nu+\overline{B}_2)+K_c^*\left[\nu\left(3\overline{A}_2\nu+2\overline{B}_2\right)+S(B_3\rho-A_3/2)-C(B_3/2+A_3\rho)+(\overline{C}_2-\overline{C}_3)\right]=0
\end{aligned}\right\} \tag{5}$$

$$\left.\begin{array}{l}
B_1 + r\overline{C}_1 + D_1 = 0 \\
A_1\rho + B_1/2 + r\overline{C}_1 = 0 \\
B_3 + r\overline{C}_3 + D_3 = 0 \\
A_3\rho + B_3/2 + r\overline{C}_3 = 0 \\
\nu(\overline{A}_2\nu + \overline{B}_2) + \overline{C}_2 = 0 \\
S(A_1 - A_3) + C(B_1 - B_3) + (\overline{C}_1 - \overline{C}_3) + (\overline{D}_1 - \overline{D}_3) = 0 \\
K_b^*(A_3S + B_3C + \overline{C}_3 + D_3) - (\overline{C}_1 + \overline{C}_3)(\rho^2 + 1/4) = 0 \\
(1/r)(\rho^2 + 1/4)(A_1S + B_1C) + 2\overline{B}_2 r = 0 \\
K_c^*\left[C(A_1\rho + B_1/2) + S(A_1/2 - B_1\rho) + (\overline{C}_1 - \overline{C}_2)\right] + 2\overline{B}_2\nu = 0 \\
-2r(3\overline{A}_2\nu + \overline{B}_2) + (1/r)(\rho^2 + 1/4)(A_3S + B_3C) = 0 \\
2\nu(3\overline{A}_2\nu + \overline{B}_2) + K_c^*\left[\nu\left(3\overline{A}_2\nu + 2\overline{B}_2\right) + S(B_3\rho - A_3/2) - C(B_3/2 + A_3\rho) + (\overline{C}_2 - \overline{C}_3)\right] = 0
\end{array}\right\} \qquad (6)$$

with respect to the dimensionless constants $A_1, B_1, \overline{C}_1 = hC_1, D_1, \overline{A}_2 = hA_2, \overline{B}_2 = hB_2, \overline{C}_2 = hC_2, A_3, B_3, \overline{C}_3 = hC_3, D_3$. It should be noted that Eqs. (5) and (6) belong to pinned and fixed supports, respectively. By setting the determination of last equations to zero, and subsequently, the critical buckling load of the non-uniform semi-rigid frames with shape factor n=2, will be obtained:

$$\det[K_i] = 0 \qquad (i = 1,2) \qquad (7)$$

The matrices $[K_i]$ (i=1,2) are given explicity in Appendix A (see Eqs. (A1) and (A2)). Similarly the corresponding matrices, $[K_i]$ (i=3,4), for the uniform frames with pinned and fixed supports, are given explicitly in Appendix A (see Eqs. (A3) and (A4)). Accordingly, the critical buckling load of the these frame, could be respectively obtained as fallows:

$$\det\left[K_i\right] = 0 \qquad (i = 3,4) \qquad (8)$$

By solving Eqs. (7) and (8), the non-dimensional critical buckling load, ρ_{cr} , is obtained, and consequently, the following critical buckling load of the frame is computed:

$$P_{cr} = \frac{\pi^2 EI_m}{(Kl_c)^2} = \begin{cases} \left(\rho_{cr}^2 + \dfrac{1}{4}\right)\dfrac{EI_c}{a^2} & for \quad n = 2 \\ \rho_{cr}^2 \dfrac{EI_c}{l_c^2} & for \quad n = 0 \end{cases} \qquad (9)$$

Moreover, Eq. (10) leads to the equivalent buckling length factor, k, of the column , which has the next value:

$$k = \frac{\pi}{\sqrt{P_{cr}^*}} \qquad (10)$$

It should be mentioned that $P^*_{cr}=P_{cr}l^2_c/EI_m$, and I_m is the moment of inertia at the middle of the column (i.e. for $x=a+0.5l_c$).

3. Parametric study

Solving numerically the buckling equations Eqs. (7) and (8), the dimensionless critical buckling load factor, ρ_{cr}, can be computed for the frame with non-uniform (n=2) and uniform (n=0) columns, respectively. This solution is valid for any desired combination of the defined non-dimensional parameters, namely the stiffness ratio ν, the dimensionless rotational stiffness of semi-rigid connections K^*_c, and the dimensionless axial stiffness of lateral elastic support K^*_b. The stiffness ratio ν varies up to 4, which is a reasonable range of beam-column characteristic properties for commonly designed steel frames. Concerning the rotational stiffness values of the connections (K_c^*), numerical results are presented for relatively low quantities $(K_c^* = 0.1,\ 0.5,\ 1.0)$, which correspond to bolted connections with low rigidity, as well as for higher ones $(K_c^* = 5,\ 10,\ \infty)$, that correspond to more rigid connections such as welded joints. Regarding the axial stiffness values of the lateral elastic support K^*_b, numerical responses are obtained for minimum value $(K_b^* = 0)$ and relatively intermediate amounts $(K_b^* = 1,\ 10)$, that correspond to un-braced and semi-braced frames, respectively, as well as for maximum ones $(K_b^* = \infty)$ that correspond to fully-braced frames. It should be added, for tapered column, the taper ratio, r, varies in the range of $0 < r \le 1$, where $r = 1$ denote a uniform member and if $r \to 0$, the member would taper to a point at the base, which is only a theoretical limit and is not practical

3.1. Uniform section (n=0)

The variation of the equivalent buckling length factor k, for the uniform frame with pinned supports, with respect to stiffness ratio v for various values of the rotational stiffness K^*_c, and various amounts of the lateral support stiffness K^*_b, are plotted in Fig. 2.

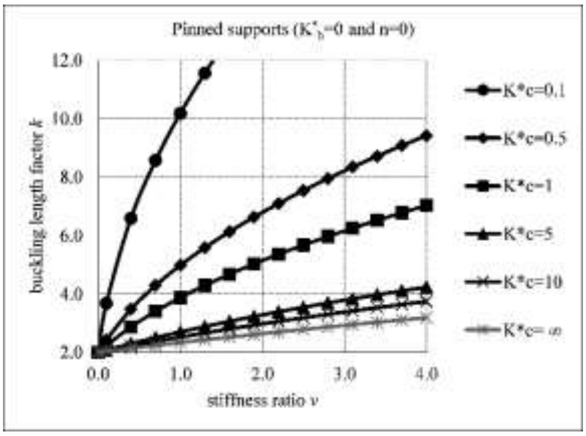
According to Fig. 2(a), in the case of the un-braced frame (i.e. $K_b^* = 0$), with pinned supports and $v \to 0$, the equivalent buckling length factor tends to $k \to 2$, irrespective of the rotational stiffness K^*_c values. This case corresponds to a pinned-fixed sway column. Also, as the stiffness ratio $v \to \infty$, the equivalent buckling length factor tends to $k \to \infty$ for all cases of K^*_c.
For intermediate values of the stiffness ratio v and low rotational stiffness amounts (i.e. K^*_c=0.1, 0.5 and 1) there is a substantial increase of the equivalent buckling length factor k, which is more pronounced when v tends to low values. This effect is reversed in the case of high rotational stiffness quantities (i.e. K^*_c=5, 10 and ∞) as v tends to high values. The same pattern pronounces also in the cases when an lateral support is present (i.e. $K_b^* \neq 0$), as shown in Figs. 2(b) up to 2(d).

In addition, regardless of the rotational stiffness K_c^* quantities, when the stiffness ratio tends to $v \to 0$, the equivalent buckling length factor tends to $k \to 1.854$, 1.243 and 0.699 for $K^*_b \to 1$, 10 and ∞, respectively. The latter case corresponds to a pinned-fixed non-sway column. Also, for all cases of K^*_c when the stiffness ratio tends to $v \to \infty$, the equivalent buckling length factor tends to $k \to 4.339$, 1.402 and 1.000 for $K^*_b \to 1$, 10 and ∞, respectively. This last case (i.e. $K^*_b \to \infty$) corresponds to a pinned-pinned non-sway column.

In Fig. 3, the same plots as above are depicted for the frame with fixed supports. More specifically, in the case of the un-braced frame (i.e. K^*_b=0) with fixed supports and $v \to 0$, the equivalent buckling length factor tends to $k \to 1$, irrespective of the rotational stiffness K^*_c values. This case corresponds to a fixed-fixed sway column. On the other hand, for $v \to \infty$, the equivalent buckling length factor tends to $k \to 2$, also regardless of the K^*_c amounts. This case corresponds to a fixed-free sway column.

For intermediate values of the stiffness ratio v and low rotational stiffness amounts (i.e. K^*_c=0.1, 0.5 and 1) there is a significant increase of the equivalent buckling length factor k which is also more pronounced as v tends to low values. The similar pattern appears also in the cases when an elastic bracing support is present (i.e. $K_b^* \neq 0$), as shown in Figs. 3(b) through 3(d).

It should be noted that regardless of the rotational stiffness K^*_c values, whenever the stiffness ratio tends to $v \to 0$, the buckling length factor tends to $k \to 0.980$, 0.843 and 0.500 for $K^*_b \to 1$, 10 and ∞, respectively. The last case (i.e. $K^*_b \to \infty$) corresponds to a fixed-fixed non-sway column. On the other hand, for $v \to \infty$, the equivalent buckling length factor tends to $k \to 1.838$, 1.237 and 0.700 for $K^*_b \to 1$, 10 and ∞, respectively. The latter case (i.e. $K^*_b \to \infty$) corresponding to a fixed-pinned non-sway column.



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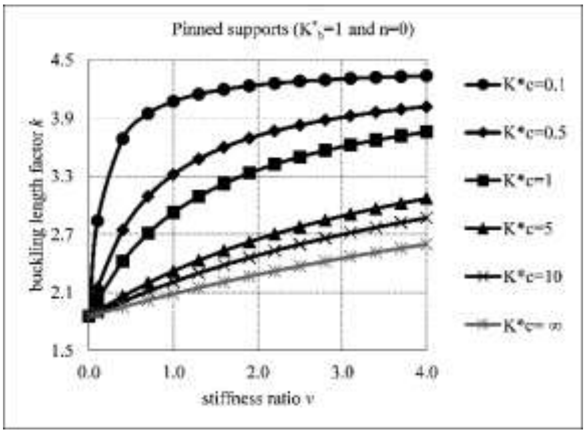

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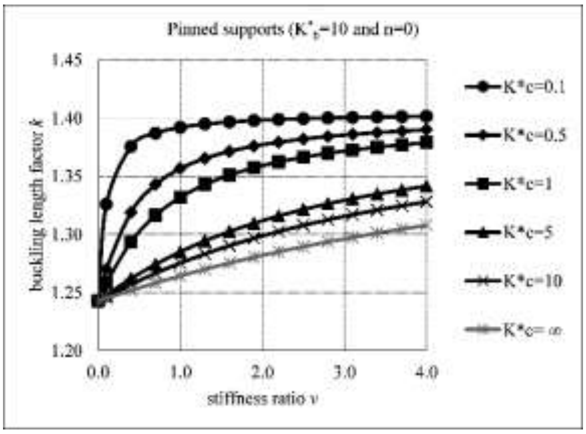

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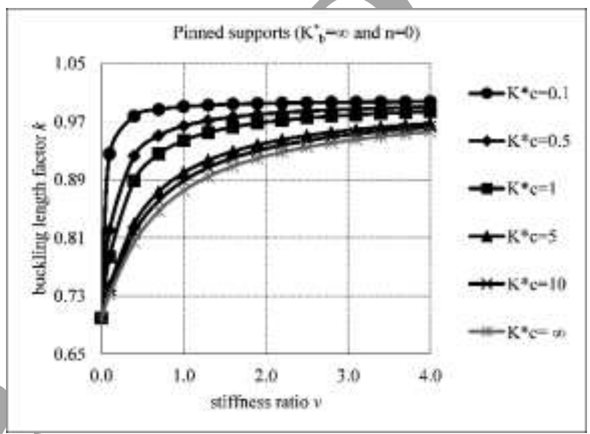
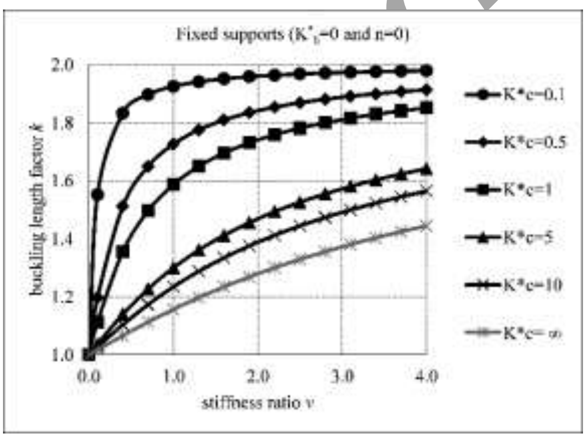
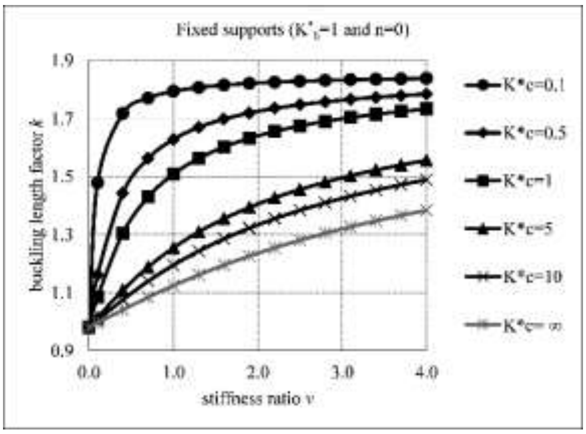


(d)

Figure 2- Buckling length factor *k* versus stiffness ratio ν for pinned support frame with prismatic columns (n=0) and various values of K^*_c and K^*_b



(a)



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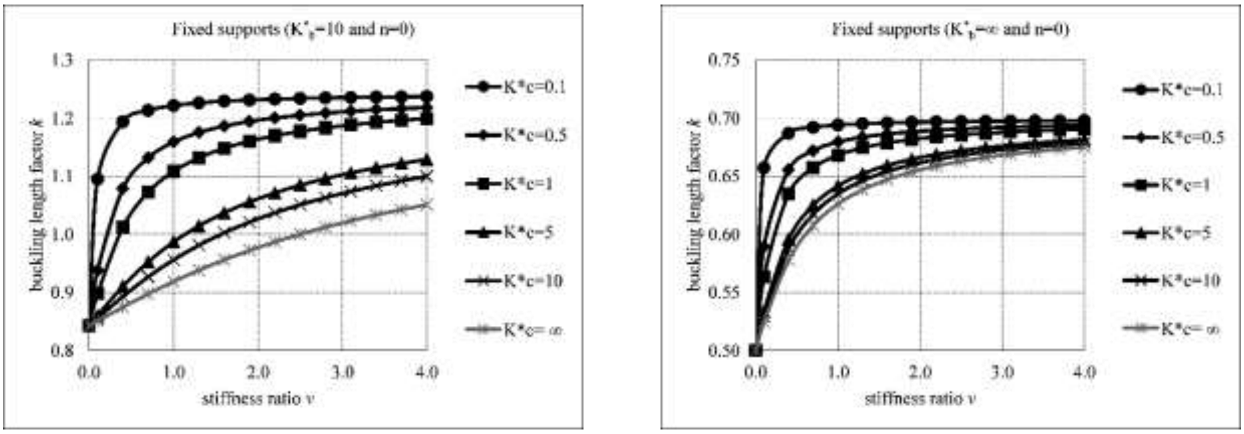


(c) **(d)**

Figure 3- Buckling length factor *k* versus stiffness ratio ν for fixed support frame with prismatic columns (n=0) and various values of K^*_c and K^*_b

3.2. Tower and open-web section (n=2)

In Fig. 4, the variation of the equivalent buckling length factor *k* , with respect to stiffness ratio *ν* for various quantities of the rotational stiffness K^*_c, and various values of the lateral support stiffness K^*_b, and *r*=1/2, are investigated for the non-uniform frame with pinned supports.

From Fig. 4(a), more specifically, in the case of the un-braced frame (i.e. $K_b^* = 0$), with pinned supports and $\nu\to0$, the equivalent buckling length factor tends to $k\to1.816$, regardless of the rotational stiffness K^*_c values. This case corresponds to a pinned-fixed sway column. Also, as the stiffness ratio $\nu\to\infty$, the equivalent buckling length factor tends to $k\to\infty$ for all cases of K^*_c. However, for the low values of the rotational stiffness in the un-braced frame, the solutions are unacceptable [3].

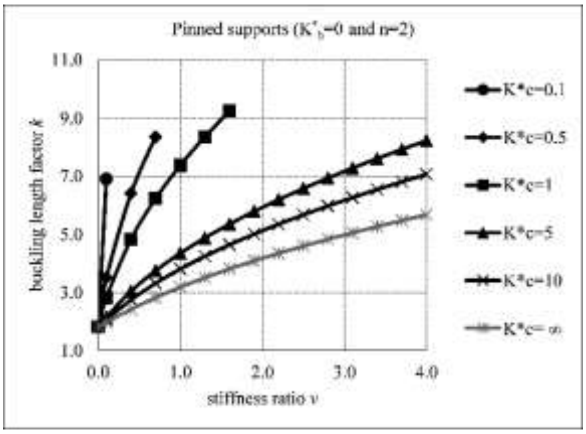
For intermediate amounts of the stiffness ratio *ν* and low rotational stiffness values (i.e. K^*_c=0.1, 0.5 and 1) there is a considerable increase of the equivalent buckling length factor *k* , which is more pronounced when *ν* tends to low quantities. This effect is reversed in the case of high rotational stiffness values (i.e. K^*_c=5, 10 and ∞) as *ν* tends to high values. The same pattern pronounces also in the cases when an lateral support is present (i.e. $K_b^* \neq 0$), as shown in Figs. 4(b) up to 4(d).

Moreover, regardless of the rotational stiffness K_c^* quantities, when the stiffness ratio tends to $\nu\to0$, the equivalent buckling length factor tends to $k\to1.810$, 1.757 and 0.726 for $K^*_b\to1$, 10 and ∞, respectively. This latter case corresponds to a pinned-fixed non-sway column. Also, for all cases of K^*_c when the stiffness ratio tends to $\nu\to\infty$, the equivalent buckling length factor tends to $k\to$na, 5.904 and 1.033 for $K^*_b\to1$, 10 and ∞, respectively. This last case (i.e. $K^*_b\to\infty$) corresponds to a pinned-pinned non-sway column.

In Fig. 5, the same plots as above are presented for the frame with fixed supports. In the case of the un-braced frame (i.e. K^*_b=0) with fixed supports and $\nu\to0$, the equivalent buckling length factor tends to $k\to1.033$, irrespective of the rotational stiffness K^*_c values. This case corresponds to a fixed-fixed sway column. On the other hand, for $\nu\to\infty$, the equivalent buckling length factor tends to $k\to2.4$, also regardless of the K^*_c amounts. This case corresponds to a fixed-free sway column.

For intermediate values of the stiffness ratio ν and low rotational stiffness amounts (i.e. K^*_c=0.1, 0.5 and 1) there is a substantial increase of the equivalent buckling length factor *k* which is also more pronounced as *ν* tends to low values. The similar pattern appears also in the cases when an elastic bracing support is present (i.e. $K_b^* \neq 0$), as shown in Figs. 5(b) through 5(d).

It should be noted that regardless of the rotational stiffness K^*_c values, whenever the stiffness ratio tends to $\nu\to0$, the buckling length factor tends to $k\to1.032$, 1.021 and 0.521 for $K^*_b\to1$, 10 and ∞, respectively. The last case (i.e. $K^*_b\to\infty$) corresponds to a fixed-fixed non-sway column. On the other hand, for $\nu\to\infty$, the equivalent buckling length factor tends to $k \rightarrow 2.383$, 2.247 and 0.726 for $K^*_b \rightarrow 1$, 10 and ∞, respectively. The latter case (i.e. $K^*_b \rightarrow \infty$) corresponding to a fixed-pinned non-sway column.



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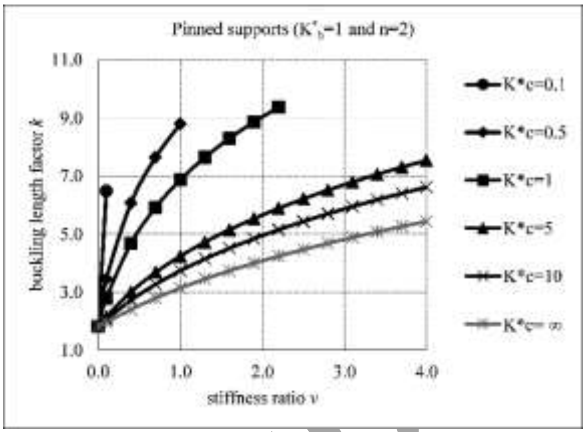

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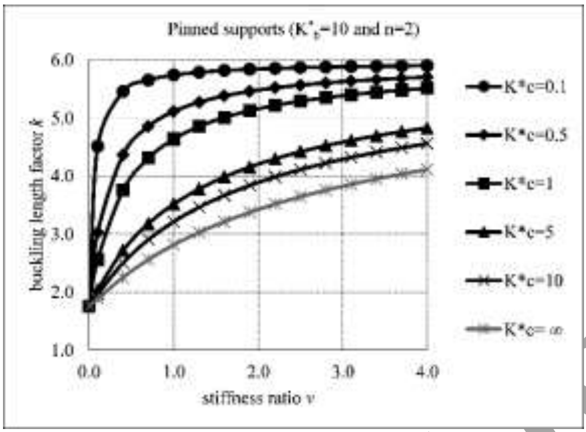

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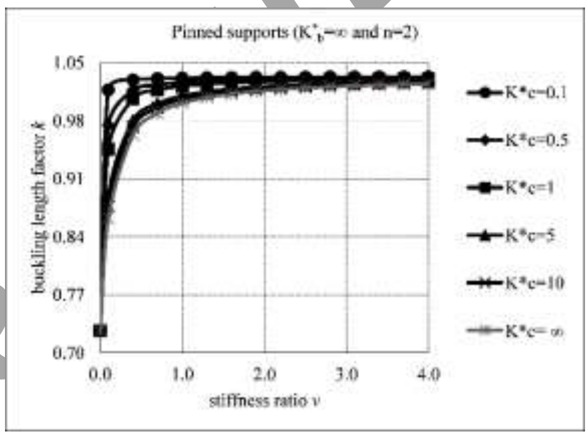
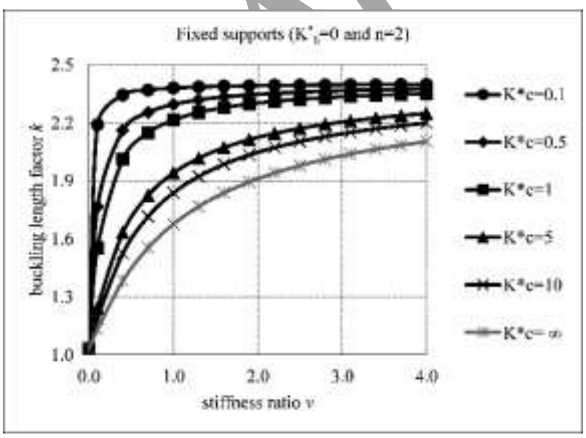
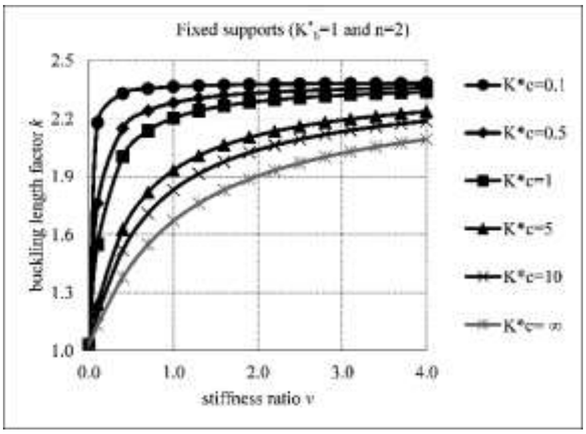


(d)

Figure 4- Buckling length factor *k* versus stiffness ratio ν for pinned support frame with non-prismatic columns (n=2) and various values of K^*_c and K^*_b (r=1/2)



(a)



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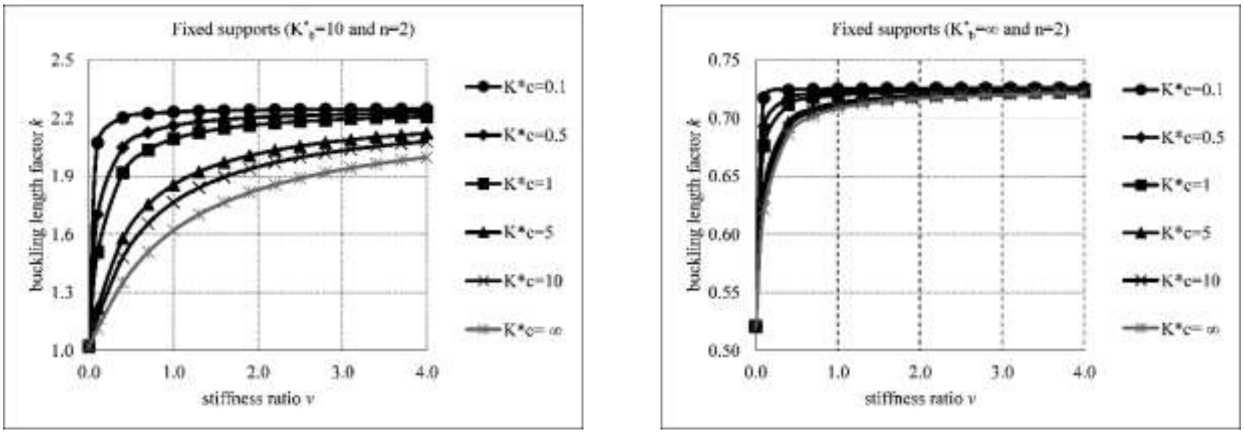

(c) **(d)**

Figure 5- Buckling length factor *k* versus stiffness ratio ν for fixed support frame with non-prismatic columns (n=2) and various values of K^*_c and K^*_b (r=1/2)

The equivalent buckling length factor, *k*, for both mentioned frames and for various values of the above parameters are presented in Tables (1) and (2). Furthermore, the results of the non-uniform frames are given by considering *r=1/2*. From these values of *k*, it is evident that the increase of the lateral stiffness K^*_b from low or zero values (corresponding to the un-braced frames) to infinity (corresponding to the fully-braced frames), will lead to a significant decrease of the equivalent buckling length factor. Consequently, the critical buckling load of the frame increased substantially. The similar pattern follows also in the presence of the rotational stiffness K^*_c. When the rotational stiffness K^*_c reduced from infinity (corresponding to the rigid connections) to very low or zero values (corresponding to the pinned connections), the buckling load of frame changed strongly. These patterns are more apparent when the shape factor increases. Furthermore, these effects are more pronounced in the case of the frame with pinned supports.

Comparing the Figs. 2 through 5 and the authors' results, a substantial reduction of the frames critical buckling load will be seen, which is caused by effects of the shape factor of the columns, lateral support, and the flexibility of joints. In addition, this reduction is declared for low values of the rotational stiffness as the stiffness ratio *ν* decreases and for high connection flexibility when *ν* increases. From Figs. 2 up to 5 one can see that the effect of the joint flexibility on the buckling load is higher in the cases of the un-braced frame, especially for the non-uniform frame, than in the braced ones.

Table 1a- Equivalent buckling length factor *k* for the uniform frame (i.e. n=0) with pinned supports and various values of ν, K^*_c and K^*_b

ν	K^*_c=0.1				K^*_c=0.5				K^*_c=1			
	K^*_b=0	K^*_b=1	K^*_b=10	K^*_b=∞	K^*_b=0	K^*_b=1	K^*_b=10	K^*_b=∞	K^*_b=0	K^*_b=1	K^*_b=10	K^*_b=∞
0.0	2.000	1.854	1.243	0.699	2.000	1.854	1.243	0.699	2.000	1.854	1.243	0.699
0.1	3.674	2.839	1.326	0.925	2.422	2.148	1.270	0.821	2.231	2.019	1.258	0.784
0.4	6.594	3.686	1.376	0.978	3.470	2.744	1.319	0.922	2.863	2.421	1.293	0.889
0.7	8.577	3.946	1.387	0.987	4.288	3.091	1.343	0.951	3.399	2.709	1.316	0.925
1.0	10.181	4.072	1.392	0.991	4.976	3.318	1.357	0.964	3.868	2.924	1.332	0.944
1.3	11.565	4.148	1.395	0.993	5.582	3.479	1.365	0.971	4.288	3.091	1.343	0.955
1.6	12.800	4.197	1.397	0.994	6.129	3.599	1.371	0.976	4.671	3.224	1.351	0.963
1.9	13.927	4.233	1.398	0.995	6.631	3.693	1.376	0.980	5.026	3.332	1.357	0.968
2.2	14.968	4.259	1.399	0.996	7.098	3.767	1.379	0.983	5.357	3.423	1.362	0.972
2.5	15.942	4.280	1.400	0.996	7.536	3.828	1.382	0.985	5.670	3.500	1.366	0.975
2.8	16.860	4.296	1.400	0.997	7.950	3.879	1.384	0.986	5.966	3.566	1.370	0.978
3.1	17.730	4.310	1.401	0.997	8.343	3.922	1.386	0.987	6.248	3.623	1.373	0.980
3.4	18.560	4.321	1.401	0.997	8.719	3.959	1.388	0.988	6.519	3.673	1.375	0.981
3.7	19.353	4.330	1.401	0.997	9.080	3.991	1.389	0.989	6.778	3.717	1.377	0.983
4.0	20.116	4.338	1.402	0.998	9.426	4.019	1.390	0.990	7.028	3.757	1.379	0.984

ν	K^*_c=5				K^*_c=10				K^*_c=∞			
	K^*_b=0	K^*_b=1	K^*_b=10	K^*_b=∞	K^*_b=0	K^*_b=1	K^*_b=10	K^*_b=∞	K^*_b=0	K^*_b=1	K^*_b=10	K^*_b=∞
0.0	2.000	1.854	1.243	0.699	2.000	1.854	1.243	0.699	2.000	1.854	1.243	0.699
0.1	2.073	1.907	1.248	0.744	2.053	1.893	1.246	0.738	2.033	1.878	1.245	0.732
0.4	2.289	2.059	1.262	0.829	2.212	2.005	1.257	0.817	2.133	1.950	1.252	0.804
0.7	2.497	2.196	1.274	0.873	2.366	2.110	1.267	0.861	2.231	2.019	1.258	0.847
1.0	2.693	2.320	1.285	0.900	2.515	2.208	1.275	0.889	2.328	2.085	1.264	0.875
1.3	2.879	2.431	1.294	0.917	2.658	2.298	1.283	0.907	2.422	2.148	1.270	0.894
1.6	3.056	2.530	1.302	0.929	2.796	2.382	1.290	0.920	2.515	2.208	1.275	0.909
1.9	3.224	2.621	1.310	0.938	2.928	2.459	1.297	0.930	2.605	2.265	1.280	0.919
2.2	3.385	2.702	1.316	0.945	3.056	2.530	1.302	0.938	2.693	2.320	1.285	0.928
2.5	3.539	2.777	1.321	0.951	3.179	2.597	1.308	0.944	2.779	2.372	1.289	0.935
2.8	3.687	2.845	1.326	0.955	3.298	2.659	1.313	0.949	2.863	2.421	1.293	0.941
3.1	3.830	2.908	1.331	0.959	3.413	2.717	1.317	0.953	2.944	2.468	1.297	0.945
3.4	3.967	2.966	1.335	0.962	3.525	2.771	1.321	0.957	3.024	2.513	1.301	0.949
3.7	4.101	3.020	1.338	0.965	3.634	2.821	1.325	0.960	3.102	2.556	1.304	0.953
4.0	4.230	3.069	1.341	0.967	3.739	2.869	1.328	0.963	3.179	2.597	1.308	0.956

Table 1b- Equivalent buckling length factor *k* for the non-uniform frame (i.e. n=2) with pinned supports and various values of ν, K^*_c and K^*_b

ν	$K^*_c=0.1$				$K^*_c=0.5$				$K^*_c=1$				$K^*_c=5$				$K^*_c=10$				$K^*_c=\infty$			
	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$
0.0	1.816	1.810	1.757	0.726	1.816	1.810	1.757	0.726	1.816	1.810	1.757	0.726	1.816	1.810	1.757	0.726	1.816	1.810	1.757	0.726	1.816	1.810	1.757	0.726
0.1	6.908	6.486	4.514	1.017	3.508	3.449	3.029	0.975	2.821	2.791	2.559	0.946	2.154	2.142	2.040	0.890	2.063	2.052	1.965	0.877	1.970	1.961	1.887	0.863
0.4	na	na	5.455	1.029	6.408	6.067	4.366	1.017	4.830	4.679	3.755	1.006	3.042	3.004	2.717	0.980	2.745	2.717	2.502	0.973	2.419	2.400	2.253	0.963
0.7	na	na	5.654	1.031	8.361	7.643	4.854	1.024	6.232	5.918	4.309	1.017	3.750	3.678	3.179	1.001	3.315	3.266	2.904	0.996	2.821	2.791	2.559	0.989
1.0	na	na	5.740	1.032	na	8.791	5.112	1.027	7.374	6.867	4.636	1.022	4.348	4.238	3.516	1.010	3.808	3.733	3.214	1.006	3.181	3.138	2.813	1.001
1.3	na	na	5.789	1.032	na	na	5.272	1.028	8.361	7.643	4.854	1.025	4.876	4.721	3.777	1.015	4.246	4.142	3.462	1.012	3.508	3.449	3.029	1.008
1.6	na	na	5.820	1.032	na	na	5.381	1.029	9.244	8.300	5.010	1.026	5.352	5.149	3.984	1.018	4.643	4.509	3.666	1.016	3.808	3.733	3.214	1.013
1.9	na	na	5.841	1.033	na	na	5.460	1.030	na	8.868	5.127	1.027	5.790	5.535	4.155	1.021	5.010	4.842	3.838	1.019	4.087	3.995	3.375	1.016
2.2	na	na	5.857	1.033	na	na	5.521	1.030	na	9.368	5.218	1.028	6.197	5.887	4.297	1.022	5.352	5.149	3.984	1.021	4.348	4.238	3.516	1.018
2.5	na	na	5.869	1.033	na	na	5.568	1.031	na	na	5.292	1.029	6.579	6.212	4.418	1.024	5.674	5.433	4.111	1.022	4.595	4.465	3.642	1.020
2.8	na	na	5.879	1.033	na	na	5.606	1.031	na	na	5.352	1.029	6.940	6.513	4.523	1.025	5.978	5.699	4.223	1.023	4.830	4.679	3.755	1.021
3.1	na	na	5.887	1.033	na	na	5.637	1.031	na	na	5.402	1.030	7.283	6.794	4.614	1.025	6.268	5.948	4.321	1.024	5.054	4.882	3.857	1.022
3.4	na	na	5.893	1.033	na	na	5.663	1.031	na	na	5.444	1.030	7.611	7.058	4.694	1.026	6.545	6.183	4.408	1.025	5.269	5.075	3.950	1.023
3.7	na	na	5.898	1.033	na	na	5.686	1.032	na	na	5.481	1.030	7.925	7.306	4.764	1.027	6.811	6.406	4.487	1.026	5.475	5.258	4.034	1.024
4.0	na	na	5.903	1.033	na	na	5.705	1.032	na	na	5.512	1.031	8.227	7.541	4.828	1.027	7.066	6.617	4.557	1.026	5.674	5.433	4.111	1.025

Table 2a- Equivalent buckling length factor *k* for the uniform frame (i.e. n=0) with fixed supports and various values of ν, K^*_c and K^*_b

ν	$K^*_c=0.1$				$K^*_c=0.5$				$K^*_c=1$				$K^*_c=5$				$K^*_c=10$				$K^*_c=\infty$			
	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$
0.0	1.000	0.980	0.843	0.500	1.000	0.980	0.843	0.500	1.000	0.980	0.843	0.500	1.000	0.980	0.843	0.500	1.000	0.980	0.843	0.500	1.000	0.980	0.843	0.500
0.1	1.553	1.478	1.095	0.657	1.197	1.161	0.938	0.590	1.113	1.084	0.897	0.563	1.037	1.014	0.860	0.533	1.027	1.005	0.856	0.529	1.017	0.995	0.851	0.524
0.4	1.834	1.718	1.194	0.687	1.512	1.443	1.079	0.656	1.356	1.304	1.012	0.635	1.139	1.108	0.910	0.596	1.104	1.075	0.893	0.587	1.066	1.041	0.875	0.578
0.7	1.898	1.771	1.213	0.692	1.650	1.562	1.132	0.672	1.497	1.429	1.073	0.657	1.227	1.188	0.953	0.625	1.173	1.139	0.927	0.617	1.113	1.084	0.897	0.608
1.0	1.926	1.794	1.222	0.694	1.727	1.628	1.159	0.679	1.587	1.508	1.108	0.668	1.300	1.254	0.987	0.642	1.234	1.195	0.956	0.635	1.157	1.124	0.919	0.626
1.3	1.942	1.807	1.226	0.695	1.776	1.670	1.175	0.684	1.650	1.562	1.132	0.674	1.361	1.309	1.014	0.652	1.288	1.243	0.981	0.646	1.197	1.161	0.938	0.638
1.6	1.953	1.815	1.229	0.696	1.810	1.698	1.186	0.686	1.696	1.602	1.148	0.679	1.412	1.355	1.037	0.659	1.335	1.285	1.003	0.654	1.234	1.195	0.956	0.647
1.9	1.960	1.821	1.231	0.696	1.835	1.719	1.194	0.688	1.731	1.632	1.160	0.682	1.456	1.393	1.056	0.665	1.376	1.322	1.021	0.660	1.269	1.226	0.972	0.654
2.2	1.965	1.826	1.233	0.697	1.855	1.735	1.200	0.690	1.759	1.656	1.170	0.684	1.494	1.427	1.071	0.669	1.412	1.355	1.037	0.664	1.300	1.254	0.987	0.659
2.5	1.969	1.829	1.234	0.697	1.870	1.748	1.205	0.691	1.782	1.675	1.177	0.686	1.526	1.455	1.085	0.672	1.445	1.384	1.051	0.668	1.329	1.280	1.000	0.663
2.8	1.972	1.832	1.235	0.697	1.882	1.758	1.209	0.692	1.801	1.690	1.183	0.687	1.555	1.481	1.096	0.675	1.474	1.409	1.063	0.671	1.356	1.304	1.012	0.666
3.1	1.975	1.834	1.236	0.698	1.892	1.766	1.212	0.692	1.817	1.704	1.188	0.688	1.581	1.503	1.106	0.677	1.500	1.432	1.074	0.673	1.381	1.327	1.023	0.669
3.4	1.977	1.835	1.236	0.698	1.901	1.773	1.214	0.693	1.830	1.715	1.193	0.689	1.603	1.522	1.114	0.679	1.524	1.453	1.083	0.675	1.403	1.347	1.033	0.671
3.7	1.979	1.837	1.237	0.698	1.908	1.779	1.216	0.693	1.842	1.724	1.196	0.690	1.624	1.540	1.122	0.680	1.545	1.472	1.092	0.677	1.425	1.366	1.042	0.673
4.0	1.980	1.838	1.237	0.698	1.914	1.784	1.218	0.694	1.852	1.733	1.199	0.691	1.642	1.556	1.129	0.681	1.565	1.489	1.100	0.679	1.445	1.384	1.051	0.675

Table 2b- Equivalent buckling length factor *k* for the non-uniform frame (i.e. n=2) with fixed supports and various values of ν, K^*_c and K^*_b

ν	$K^*_c=0.1$				$K^*_c=0.5$				$K^*_c=1$				$K^*_c=5$				$K^*_c=10$				$K^*_c=\infty$			
	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$	$K^*_b=0$	$K^*_b=1$	$K^*_b=10$	$K^*_b=\infty$
0.0	1.033	1.032	1.021	0.521	1.033	1.032	1.021	0.521	1.033	1.032	1.021	0.521	1.033	1.032	1.021	0.521	1.033	1.032	1.021	0.521	1.033	1.032	1.021	0.521
0.1	2.191	2.178	2.072	0.717	1.767	1.760	1.702	0.693	1.554	1.549	1.508	0.676	1.243	1.241	1.219	0.640	1.189	1.187	1.169	0.632	1.132	1.130	1.115	0.621
0.4	2.345	2.329	2.202	0.724	2.161	2.148	2.046	0.717	2.012	2.002	1.918	0.711	1.632	1.626	1.579	0.696	1.524	1.520	1.481	0.692	1.382	1.379	1.350	0.686
0.7	2.370	2.354	2.223	0.725	2.254	2.240	2.126	0.721	2.149	2.136	2.036	0.717	1.825	1.818	1.754	0.708	1.716	1.709	1.655	0.705	1.554	1.549	1.508	0.702
1.0	2.381	2.365	2.232	0.725	2.296	2.281	2.161	0.722	2.215	2.202	2.093	0.720	1.941	1.932	1.856	0.713	1.838	1.830	1.765	0.711	1.676	1.670	1.620	0.708
1.3	2.387	2.370	2.236	0.725	2.319	2.304	2.181	0.723	2.254	2.240	2.126	0.721	2.017	2.007	1.923	0.716	1.923	1.914	1.840	0.715	1.767	1.760	1.702	0.712
1.6	2.390	2.374	2.239	0.726	2.335	2.319	2.193	0.724	2.280	2.265	2.147	0.722	2.072	2.062	1.971	0.718	1.986	1.976	1.895	0.717	1.838	1.830	1.765	0.715
1.9	2.393	2.376	2.241	0.726	2.346	2.330	2.202	0.724	2.298	2.283	2.163	0.723	2.114	2.102	2.006	0.719	2.034	2.024	1.937	0.718	1.894	1.886	1.815	0.717
2.2	2.394	2.378	2.243	0.726	2.353	2.338	2.209	0.724	2.312	2.297	2.174	0.723	2.146	2.134	2.034	0.720	2.072	2.062	1.971	0.719	1.941	1.932	1.856	0.718
2.5	2.396	2.379	2.244	0.726	2.360	2.344	2.214	0.725	2.322	2.307	2.183	0.724	2.172	2.159	2.056	0.721	2.104	2.092	1.997	0.720	1.979	1.969	1.889	0.719
2.8	2.397	2.380	2.245	0.726	2.364	2.348	2.218	0.725	2.331	2.315	2.190	0.724	2.193	2.180	2.074	0.721	2.129	2.118	2.020	0.721	2.012	2.002	1.918	0.720
3.1	2.398	2.381	2.246	0.726	2.368	2.352	2.221	0.725	2.338	2.322	2.196	0.724	2.210	2.197	2.089	0.722	2.151	2.139	2.038	0.721	2.039	2.029	1.942	0.720
3.4	2.398	2.382	2.246	0.726	2.371	2.355	2.224	0.725	2.343	2.328	2.201	0.724	2.225	2.212	2.102	0.722	2.170	2.157	2.054	0.722	2.064	2.053	1.963	0.721
3.7	2.399	2.382	2.247	0.726	2.374	2.358	2.226	0.725	2.348	2.333	2.205	0.724	2.238	2.225	2.112	0.723	2.186	2.173	2.068	0.722	2.085	2.074	1.981	0.721
4.0	2.400	2.383	2.247	0.726	2.377	2.360	2.228	0.725	2.352	2.337	2.208	0.725	2.249	2.235	2.122	0.723	2.200	2.187	2.080	0.722	2.104	2.092	1.997	0.722

4. Conclusions

Based on a developed buckling analysis portal frame, the paper has first presented a closed-form expression for computing the "exact" critical load and corresponding equivalent buckling length of non-uniform frames with semi-rigid connections and elastic bracing system. The methodology is based on the exact solution of the governing differential equations for buckling of the uniform and non-uniform frames. Also, the proposed formulation can be used for the tapered column with various end boundary conditions in the particular cases. From the parametric and numerical solution point of view, the following results are concluded:

1. The combined effect of the shape factor, taper ratio, elastic bracing system, and joint flexibility on the critical buckling load and corresponding equivalent buckling length factor of portal steel frames is very significant. As a result, these effects should be considered in design of such structures.
2. The connection flexibility will reduce critical buckling load of the frame. Consequently, it increases the corresponding equivalent buckling length factor. These effects are similar to the elastic bracing system.
3. The equivalent buckling length factor of the non-uniform portal frames will increase, when the shape factor as well as stiffness ratio increases. For the un-braced frames with pinned supports, this effect is even more pronounced.

Appendix A

The unknown constants matrixes, $\mathbf{K_1}$ and $\mathbf{K_2}$, for the fixed and pinned supports' frame with shape factor n = 2, respectively, have the next values:

$$K_3 = \begin{bmatrix} 0 & 1 & r & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho & 1/2 & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & r & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 1/2 & r & 0 \\ 0 & 0 & 0 & 0 & \nu^2 & \nu & 1 & 0 & 0 & 0 & 0 \\ S & C & 1 & 1 & 0 & 0 & 0 & -S & -C & -1 & -1 \\ 0 & 0 & -(\rho^2+1/4) & 0 & 0 & 0 & 0 & K_b^* S & K_b^* C & K_b^*-(\rho^2+1/4) & K_b^* \\ (\rho^2+1/4)S & (\rho^2+1/4)C & 0 & 0 & 0 & 2r^2 & 0 & 0 & 0 & 0 & 0 \\ K_c^*(S/2+\rho C) & K_c^*(C/2-\rho S) & K_c^* & 0 & 0 & 2\nu & -K_c^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6\nu r^2 & -2r^2 & 0 & (\rho^2+1/4)S & (\rho^2+1/4)C & 0 & 0 \\ 0 & 0 & 0 & 0 & 3\nu^2(K_c^*+2) & 2\nu(K_c^*+1) & K_c^* & -K_c^*(S/2+\rho C) & -K_c^*(C/2-\rho S) & -K_c^* & 0 \end{bmatrix} \quad \text{(A1)}$$

$$K_4 = \begin{bmatrix} 0 & 1 & r & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\rho^2+1/4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & r & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\rho^2+1/4) & 0 & 0 \\ 0 & 0 & 0 & 0 & \nu^2 & \nu & 1 & 0 & 0 & 0 & 0 \\ S & C & 1 & 1 & 0 & 0 & 0 & -S & -C & -1 & -1 \\ 0 & 0 & -(\rho^2+1/4) & 0 & 0 & 0 & 0 & K_b^* S & K_b^* C & K_b^*-(\rho^2+1/4) & K_b^* \\ (\rho^2+1/4)S & (\rho^2+1/4)C & 0 & 0 & 0 & 2r^2 & 0 & 0 & 0 & 0 & 0 \\ K_c^*(S/2+\rho C) & K_c^*(C/2-\rho S) & K_c^* & 0 & 0 & 2\nu & -K_c^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6\nu r^2 & -2r^2 & 0 & (\rho^2+1/4)S & (\rho^2+1/4)C & 0 & 0 \\ 0 & 0 & 0 & 0 & 3\nu^2(K_c^*+2) & 2\nu(K_c^*+1) & K_c^* & -K_c^*(S/2+\rho C) & -K_c^*(C/2-\rho S) & -K_c^* & 0 \end{bmatrix} \quad \text{(A2)}$$

The following parameters are used in the last matrices:

$$\rho^2 = \frac{Pa^2}{EI_c} - \frac{1}{4}, \quad \nu = \frac{I_c l_b}{I_b h}, \quad r = \frac{a}{h}, \quad K_b^* = \frac{K_b h^3}{EI_c} r^2, \quad K_c^* = \frac{K_c l_b}{EI_b}, \quad S = \sqrt{\frac{1}{r}} \sin\left(\rho \ln(\frac{1}{r})\right),$$

$$C = \sqrt{\frac{1}{r}} \cos\left(\rho \ln(\frac{1}{r})\right).$$

The unknown constants matrixes, $\mathbf{K_3}$ and $\mathbf{K_4}$, for the fixed and pinned supports' frame with shape factor n = 0, respectively are defined by the next relationships:

$$
K_5 = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\rho & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \nu^2 & \nu & 1 & 0 & 0 & 0 & 0 \\
\sin\rho & \cos\rho & 1 & 1 & 0 & 0 & 0 & -\sin\rho & -\cos\rho & -1 & -1 \\
0 & 0 & -\rho^2 & 0 & 0 & 0 & 0 & K_b^*\sin\rho & K_b^*\cos\rho & K_b^*-\rho^2 & K_b^* \\
\rho^2\sin\rho & \rho^2\cos\rho & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
K_c^*\rho\cos\rho & -K_c^*\rho\sin\rho & K_c^* & 0 & 0 & 2\nu & -K_c^* & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -6\nu & -2 & 0 & \rho^2\sin\rho & \rho^2\cos\rho & 0 & 0 \\
0 & 0 & 0 & 0 & 3\nu^2(K_c^*+2) & 2\nu(K_c^*+1) & K_c^* & -K_c^*\rho\cos\rho & K_c^*\rho\sin\rho & -K_c^* & 0
\end{bmatrix} \quad \text{(A3)}
$$

$$
K_6 = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \nu^2 & \nu & 1 & 0 & 0 & 0 & 0 \\
\sin\rho & \cos\rho & 1 & 1 & 0 & 0 & 0 & -\sin\rho & -\cos\rho & -1 & -1 \\
0 & 0 & -\rho^2 & 0 & 0 & 0 & 0 & K_b^*\sin\rho & K_b^*\cos\rho & K_b^*-\rho^2 & K_b^* \\
\rho^2\sin\rho & \rho^2\cos\rho & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
K_c^*\rho\cos\rho & -K_c^*\rho\sin\rho & K_c^* & 0 & 0 & 2\nu & -K_c^* & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -6\nu & -2 & 0 & \rho^2\sin\rho & \rho^2\cos\rho & 0 & 0 \\
0 & 0 & 0 & 0 & 3\nu^2(K_c^*+2) & 2\nu(K_c^*+1) & K_c^* & -K_c^*\rho\cos\rho & K_c^*\rho\sin\rho & -K_c^* & 0
\end{bmatrix} \quad \text{(A4)}
$$

in which, $\rho^2 = \dfrac{Pl_c^2}{EI_c}, \quad \nu = \dfrac{I_c l_b}{I_b l_c}, \quad K_b^* = \dfrac{K_b l_c^3}{EI_c}, \quad K_c^* = \dfrac{K_c l_b}{EI_b}.$

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