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Analytical Solution for Influence of Yield Stress on Two Phase Stratified Flow through the Pipe

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Abstract
For two immiscible fluids in steady state, stratified laminar, fully developed flow which one of them is Newtonian and the other is Bingham plastic the motion equations in horizontal pipe with appropriate boundary condition have been solved analytically. Velocity distribution for two phases and the location of plug region related to Bingham plastic fluid have been reported. The results show that the non-Newtonian rheological properties have negligible effects on two phase velocity profile.

Keywords: Analytical, stratified, two-phase, Bingham plastic, non-Newtonian

1. Introduction

The simultaneous flow of two immiscible fluid, liquid-liquid or gas-liquid, in a pipe is commonly found in the petroleum and chemical processing industries, in steam generation equipment, and in nuclear reactors.

Stratified flow is a basic flow configuration in horizontal and inclined pipes. Due to the density difference between the two phases, they tend to segregate. Many pipeline systems are designed to operate in stratified flow region and many of process fluids have non-Newtonian behavior. Many works have been studied in two phase flow through pipes analytically and numerically that liquid has Newtonian behavior [1-4]. The annular flow of a lubricant as Bingham plastic non-Newtonian fluid model in contact with a Newtonian fluid has been studied in [5]. However, some information is available for the cases when the liquid phase is non-Newtonian which mostly are concerned to numerical or experimental investigations or single phase [6-13]. Here exact solutions are obtained for two phase Newtonian-Bingham plastic fluid that has not been reported yet.

2. Mathematical modeling

Figure 1 illustrates two phase stratified flow and plug region in pipe and appropriate bipolar coordinate for this model. The appropriate coordinate system for solving the flow

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problem, for stratified flow is the well-known bipolar coordinate system. Coordinate φ represents the view angle of the two poles from an arbitrary point in the flow domain, coordinate x relates to the ratio of the radius vectors r_1 and r_2 $\xi = \ln\left(\frac{r_1}{r_2}\right)$. The pipe perimeter and the interface between the fluids are isolines with constant coordinates φ, so that the upper section of the tube wall bounding the lighter phase is represented by φ_1, while the bottom of pipe, bounding the denser phase, is represented by φ_2. The interface coincides with the curve of φ_i. The transformation functions for Cartesian and bipolar coordinate are:

$$x = \frac{R\sin\varphi_1 \sinh\xi}{\cosh\xi - \cos\varphi}, \qquad y = \frac{R\sin\varphi_1 \sin\xi}{\cosh\xi - \cos\varphi} \tag{1}$$

The steady-state, laminar fully developed flow which two phases are immiscible, homogeneous and incompressible and all body forces are negligible, the lighter phase is Newtonian fluid and denser phase is non-Newtonian fluid that obeys from Bingham plastic model. The motion equation for upper and lower phases can be simplified respectively as follows:

$$\left[\frac{\cosh\xi - \cos\varphi}{R\sin\varphi_1}\right]^2 \left(\frac{\partial^2 v_1}{\partial\varphi^2} + \frac{\partial^2 v_1}{\partial\xi^2}\right) = -\frac{1}{\mu_1}\frac{dp}{dz} \tag{2}$$

$$\left[\frac{\cosh\xi - \cos\varphi}{R\sin\varphi_1}\right]^2 \left(\frac{\partial\tau_{\varphi z}}{\partial\varphi}\right) - \frac{\tau_{\varphi z}}{R}\sin\varphi = -\frac{dp}{dz} \tag{3}$$

The non-Newtonian phase is considered as a very thin film, therefore the velocity variation in x direction has been neglected. The Bingham fluid model is defined as follows [14]:

$$\tau_{\varphi z} = \begin{cases} \left(\mu_0 + \frac{\tau_0}{|\dot\gamma|}\right)\dot\gamma_{\varphi z}, & |\tau| \ge \tau_0 \\ \dot\gamma_{\varphi z} = 0, & |\tau| < \tau_0 \end{cases} \tag{4}$$

The Bingham plastic model takes into account two parameters, the yield stress t_0, and the plastic viscosity m_0, to fully characterize the material rheology. Note that once the fluid flows, the plastic viscosity defines the rate of change of the excess shear stress $t - t_0$ with the shear rate. $|\dot\gamma|$ and $|\tau|$ are rate of strain and deviatoric stress second invariants respectively and are defined by

$$|\dot{\gamma}| = \sqrt{\frac{1}{2}(\dot{\gamma}:\dot{\gamma})} = \left[\frac{1}{2}\sum_{i,j=1}^{2}\left[\dot{\gamma}_{ij}\right]^2\right]^{1/2} \qquad |\tau| = \sqrt{\frac{1}{2}(\tau:\tau)} = \left[\frac{1}{2}\sum_{i,j=1}^{2}\left[\tau_{ij}\right]^2\right]^{1/2} \tag{5}$$

Where the shear rate tensor is:

$$\dot{\gamma} = \left(\nabla v + \nabla v^{\dagger}\right)$$

The boundary conditions for solving the above equations are:

1. No-slip condition at the upper and lower walls

$$v_1\big|_{\varphi=\varphi_1} = 0, \qquad v_2\big|_{\varphi=\varphi_1+\pi} = 0 \tag{6}$$

2. No-slip condition at the triple point which are a part of wall (where the interface meets the tube wall)

$$v_1\big|_{\xi=\pm\infty} = 0 \tag{7}$$

3. Continuity of velocities across the interface of two phases

$$v_1\big|_{\varphi=\varphi_i} = v_2\big|_{\varphi=\varphi_i} \tag{8}$$

4. Plug core in the Bingham fluid phase

$$\left.\frac{\partial v_2}{\partial \varphi}\right|_{\varphi=\varphi_p} = 0 \tag{9}$$

Where φ_p presents the plug region location in non-Newtonian phase.

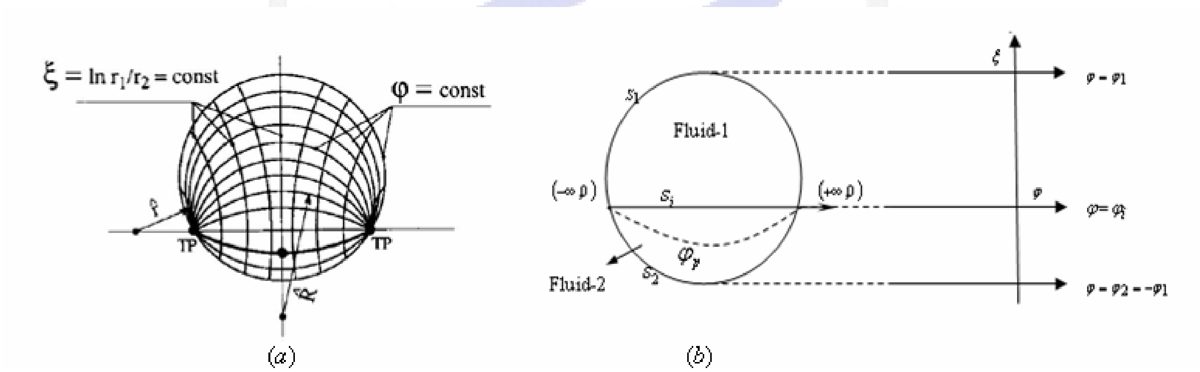

Fig. 1 Schematic representation of (a) bipolar coordinate, (b) stratified two phase flow

2.1. The Newtonian Phase Velocity Profile

Because of non-homogenous term in Newtonian phase motion equation Eq. (2), the general solution v_1, is composed of particular solution v_1^p, and the homogenous solution v_1^h:

$$v_1 = v_1^p + v_1^h \tag{10}$$

Where the particular solution is as follow [2]:

$$v_1^p = \frac{R^2 \sin(\varphi_1)}{2\mu_1} \frac{\sin(\varphi - \varphi_1)}{(\cosh\xi - \cos\varphi)} \tag{11}$$

From substitution of Eq. (10) into Eq. (2), the homogenous partial differential equation and corresponding boundary conditions can be derived as follows:

$$\frac{\partial^2 v_1^h}{\partial \varphi^2} + \frac{\partial^2 v_1^h}{\partial \xi^2} = 0 \tag{12}$$

$$v_1^h\Big|_{\xi=\pm\infty} = 0 \tag{13}$$

$$v_1^h\Big|_{\varphi=\varphi_i} + \frac{R^2 \sin(\varphi_0)}{2\mu_1} \frac{\sin(\varphi_i - \varphi_0)}{(\cosh\xi - \cos\varphi_i)} = v_2\Big|_{\varphi=\varphi_i} \tag{14}$$

$$v_1^h\Big|_{\varphi=\varphi_1} = 0 \tag{15}$$

The solution of homogenous part of the first phase equation can be obtained in the form of Fourier integral:

$$v_1^h = \int_0^\infty \phi(\omega) \sinh\left[\omega(\varphi - \varphi_1)\right] \cos(\omega\xi)\, d\omega \tag{16}$$

Obviously this solution satisfies the Eqs. (12) and (15). Boundary condition Eq. (13) is an underlying requirement for the legitimate use of Fourier integral in obtaining the solution. Substituting the homogenous solution Eq. (16) in Eq. (14) yields:

$$\int_0^\infty \phi(\omega) \sinh\left[\omega(\varphi_i - \varphi_1)\right] \cos(\omega\xi)\, d\omega + \frac{R^2 \sin(\varphi_0)}{2\mu_1} \frac{\sin(\varphi_i - \varphi_0)}{(\cosh\xi - \cos\varphi_i)} = v_2\Big|_{\varphi=\varphi_i} \tag{17}$$

The Eq. (17) can be simplified:

$$\int_0^\infty \tilde{\phi}(\omega) \cos(\omega\xi)\, d\omega = 1 \tag{18}$$

Where

$$\tilde{\phi}(\omega) = \frac{\phi(\omega) \sinh\left[\omega(\varphi_i - \varphi_1)\right]}{B_1}, \quad B_1 = v_2\Big|_{\varphi=\varphi_i} - \frac{R^2 \sin(\varphi_0)}{2\mu_1} \frac{\sin(\varphi_i - \varphi_0)}{(\cosh\xi - \cos\varphi_i)} \tag{19}$$

In order to obtain the $\tilde{f}(w)$, the following integral is employed [2]:

$$\int_0^\infty \frac{2\omega(\cosh\xi-\cos\varphi)}{\sinh(\pi\omega)}\cos(\omega\xi)d\omega=1$$

(20)

By comparing (18) and (20)

$$\varphi^*(\omega)=\frac{2B_1^*\omega(\cosh\xi-\cos\varphi)}{\sinh(\pi\omega)\sinh\left[\omega(\varphi_i-\varphi_1)\right]} \tag{21}$$

Finally the dimensionless velocity distribution for upper phase is:

$$v_1^*(\xi,\varphi)=\int_0^\infty \phi^*(\omega)\sinh\left[\omega(\varphi-\varphi_1)\right]\cos(\omega\xi)d\omega+2\sin(\varphi_1)\frac{\sin(\varphi-\varphi_1)}{(\cosh\xi-\cos\varphi)} \tag{22}$$

Where superscript * shows the term has been non dimensionlized by $\frac{R^2}{4\mu_1}\frac{dp}{dz}$.

2.2. The Bingham Plastic Phase Velocity Profile

The lower phase momentum equation Eq. (3) is an ordinary differential equation and the integral constants can be obtained by using of boundary conditions, Eqs. (7) and (8). The dimensionless velocity distribution of non-Newtonian phase can be derived:

$$v_2^*(\varphi)=\begin{cases}C_1(\varphi-\varphi_1-\pi)+\tilde{\mu}\left(L(\varphi)-L(\varphi_1+\pi)\right) & \xi=0\\ C_2(\varphi-\pi-\varphi_1)+2\tilde{\mu}\left(H(\varphi_1+\pi)-H(\varphi)\right) & \xi\neq0\end{cases} \tag{23}$$

Where

$$C_1=\tilde{m}\left\{\frac{2}{3}\sin^2 j_1\left[\frac{3}{\tan\left(\frac{j_p}{2}\right)}-\frac{1}{\tan^3\left(\frac{j_p}{2}\right)}\right]+Bi\left[\sin j_1\left(\frac{1}{1-\cos j_p}\right)\right]\right\} \tag{24}$$

$$C_2=2\tilde{\mu}\left(Bi\sin\varphi_1\left(\frac{1}{\cosh\xi-\cos\varphi_p}\right)+2\sin^2\varphi_1 K(\varphi_p)\right) \tag{25}$$

$$\begin{aligned} H(\varphi) = & \operatorname{Bi}\sin\varphi_1\left(\frac{2}{\sqrt{\cosh^2\xi-1}}\arctan\left(\frac{(\cosh\xi+1)\tan\left(\frac{\varphi}{2}\right)}{\sqrt{\cosh^2\xi-1}}\right)\right) \\ & +2\sin^2\varphi_1\left[\frac{2I\left(\frac{1}{(\cosh\xi-1)}+1\right)}{(\cosh\xi+1)\sqrt{\cosh^2\xi-1}}+\frac{\ln(\cos\varphi-\cosh\xi)}{\cosh^2\xi-1}\right] \end{aligned} \tag{26}$$

$$\begin{aligned} K(\varphi) = & \frac{2\tan\frac{\varphi}{2}}{(\cosh^2\xi-1)\left(\cosh^2\xi\tan^2\left(\frac{\varphi}{2}\right)+\tan^2\left(\frac{\varphi}{2}\right)+\cosh\xi-1\right)}+ \\ & 2\cosh\xi\left(\cosh^2\xi-1\right)^{-\frac{3}{2}}\arctan\left((\cosh\xi+1)\frac{\tan\frac{\varphi}{2}}{\sqrt{\cosh^2\xi-1}}\right) \end{aligned} \tag{27}$$

$$L(\varphi)=\frac{2}{3}\sin^2\varphi_1\left[4\ln\left(\tan\frac{\varphi}{2}\right)-\tan^2\left(\frac{\varphi}{2}\right)-2\ln\left(1+\tan^2\frac{\varphi}{2}\right)\right]+2\operatorname{Bi}\left[\sin\varphi_1\tan\left(\frac{\varphi}{2}\right)\right] \tag{28}$$

$$v_2^*=\frac{v_2}{\frac{R^2}{4\mu_1}\frac{dp}{dz}},\quad \tilde{\mu}=\frac{\mu_1}{\mu_0},\quad \operatorname{Bi}=\frac{\tau_0}{\frac{R}{2}\frac{dp}{dz}},\quad I=\int\arctan\left(\frac{(\cosh\xi+1)\tan\left(\frac{\varphi}{2}\right)}{\sqrt{\cosh^2\xi-1}}\right)d\varphi \tag{29}$$

The Bingham number Bi, is a dimensionless number, physically represents the ratio of yield stress to viscous stress. The application of dimensionless velocity profile Eq. (23), is limited by one of above mentioned simplification.

A Bingham fluid does not deform until the stress level reaches the yield stress, after which the “excess stress” above the yield stress drives the deformation. This results in a two-layered flow consisting of a ‘plug layer’ and a ‘shear layer’. For single phase Bingham plastic flow through the pipe, the plug region radius can be calculated [8]:

$$r_p=\frac{2\tau_0 L}{\Delta p},\qquad \frac{r_p}{R}=\operatorname{Bi} \tag{30}$$

3. Result and Conclusion

Newtonian and non-Newtonian two phase stratified flow in pipes in the case of steady-state, laminar, fully-developed, incompressible and immiscible fluids have been studied analytically and the velocity profiles by assuming a thin layer of non-Newtonian phase, which the velocity variation can be ignored in x direction, are reported. The results show that, the non-Newtonian fluid rheological properties affect the two-phase velocity profiles but for the case of gas and non-Newtonian, this effect isn't noticeable in gas phase whereas it has significant effect on non-Newtonian phase.

Figure (2) shows the variation of dimensionless velocity for lower phase in a constant pressure gradient at ξ=0, rheological properties affect the plug region, as expected, plug region increases by increasing the Bingham number. Figure (3) represents the variation of the two phase velocity with $\tilde{m}$ at ξ=0, the main effect of $\tilde{m}$ is obtained for $\tilde{m} > 0.1$.

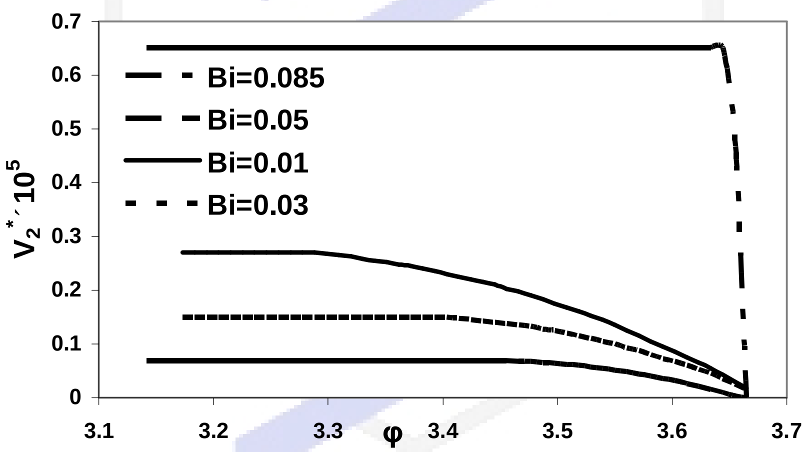

FIG. 2, Variation of the non-Newtonian phase velocity with the Bingham number

$$\tilde{\mu} = 0.1, \varphi_1 = \frac{\pi}{6}, \xi = 0$$

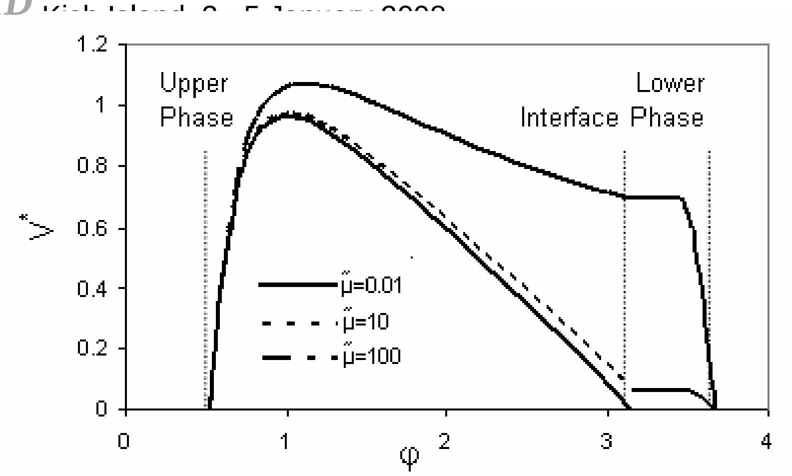

FIG. 3, Variation of the two phase velocity with the $\tilde{m}$, $Bi = 0.05, \varphi_1 = \frac{\pi}{6}, \xi = 0$

Nomenclature

A Flow area [m^2]

Bi Bingham number

D Pipe diameter [m]

h Lower layer Depth [m]

L Pipe length [m]

p Pressure [pa]

Q Flow rate [m^3/s]

R Pipe radius [m]

v Phase velocities [m/s]

z Axial direction

Greek Symbols

μ Dynamic viscosity [kg/m.s]

τ_0 Yield stress

τ stress [pa/s]

$\dot{\gamma}$ Shear rate [m/s^2]

ξ Ratio of the radius vectors(related to the bipolar coordinate)

φ View angle of the interface (related to the bipolar coordinate)

Subscripts

1 Upper phase

2 Lower phase

i Interface

p Plug flow

Superscripts

* Non-dimension

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