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Two-Person Games for Stochastic Network Interdiction

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ABSTRACT:

This paper presents a new formulation for stochastic network interdiction with endogenous uncertainty, in which optimization decisions can influence stochastic processes by altering the corresponding probability space. As an application, we can consider a planning problem in which a decision maker (interdictor), subject to limited resources, installs some detectors at border checkpoints in a transportation network in order to minimize the probability that a smuggler can traverse the residual network undetected or to maximize the smugglers traversing cost. The detecting probabilities are assumed to be known a priori, and installing detector decreases the likelihood of detection.

The resulting problem is a two stage stochastic program, in that, at the first stage the interdictor decides which links to install detector for interdicting the smuggler's moves by detecting him or decreasing his traversing cost. The interdictor's decision can influence the probabilities of traversing links successfully. In the second stage the smuggler seeks to choose the shortest $s-t$ path in the survival network. The resulting two stage programming problem is non-linear due to existing of scenario probabilities products. We use distribution shaping approach to handle decision-dependent probabilities. It uses a sequence of distributions, successively conditioned on the influencing decision variables, and characterizes these by linear inequalities.

KEYWORDS: interdiction problem, endogenous uncertainty, Stackelberg game, two-stage programming.

۱ INTRODUCTION

Networks are complex structures that appear in many societal, economic, and industrial systems such as telecommunication, electric power, transportation, natural gas and petroleum distribution systems. Due to the importance of network-based systems, having an effective protection plan is a vital need to decrease network vulnerability to natural disasters or intentional threats (e.g., terrorist attacks, random failures, earthquakes, etc.).

Network interdiction is an area that has caught researchers' attention for many years. The basic interdiction problem in the literature has been formulated as a Stackelberg game, which is a sequential game between two opposing players called defender and interdictor or attacker in this context. The defender tries to optimize some objective function, for instance, by finding the shortest path [۱], shipping a maximum flow [۲] , or ensuring maximum coverage of the demand in a facility interdiction problem [۳]. On the other hand, the interdictor attempts to limit the defender's achievable objective value by attacking some links to reduce the network capacity or increase the traversing time and cost. It is commonly assumed that a limited budget is available to the interdictor, which restricts the amount of potential damage to a network.

Interdiction models were introduced by the seminal work of Wollmer (۱۹۶۴). It has been proven that this problem is NP-hard for general graphs [۴]. Network interdiction problems have received considerable attention in the past decade in the literature because of their numerous applications in nuclear smuggling [۵], electrical grid analysis [۶] , drug enforcement [۷], military planning [۸] and supply chains [۹].

There exist a variety of network interdiction problems in the literature including the shortest path [۱], maximum flow [۲] and facility interdiction problems [۳]. In this paper, we focus on shortest path network interdiction; that is the problem of interdicting some arcs in a network in order to maximize the shortest path length. The defender wishes to traverse a path of minimum length (or time or cost) between two specified nodes, s and t, in a directed network (over a $s-t$ path). The interdictor wants to impose additional cost on the network user when traveling between the source and destination nodes.

The first shortest path interdiction model was introduced in [۱]. Other researchers considered the problem of finding the most vital arcs in a network which is a variation of shortest path interdiction models [۱۰]. An explicit formulation and solution procedure were introduced in [۱۱] based on adding some "supper valid inequalities". Another study considered removing the critical path instead of some critical arcs in the shortest path network interdiction problem [۱۲]. The stochastic variants of network flow interdiction were studied in [۱۳], [۵] and [۱۴].

In the classical paradigm of stochastic programming, it is assumed that decisions do not influence the probability space of the underlying stochastic processes, i.e. uncertainty is exogenous. The exogenous uncertainty assumption allows for the design of efficient solution algorithms for real-sized problems and from a practical point of view can be satisfied by a wide range of applications. However, there is an important class of decision-making problems for which stochastic programming is a suitable modeling approach, that violates this assumption and can consequently be very challenging to solve. In this class of stochastic programs, decisions can influence the stochastic process and, therefore, uncertainty is endogenous.

This paper is concerned with stochastic programs under endogenous uncertainty in which optimization decisions can influence stochastic processes by altering the corresponding probability space. This point of view is never applied in network interdiction problem in the literature.

۲ Problem description and formulations

In this section, we describe a conventional bilevel formulation of shortest path network interdiction problem and the existing trilevel interdiction models.

۲,۱ Bilevel shortest path network interdiction

We explain the mathematical notation used throughout the paper, in this section. Given a weighted directed graph $G(N,A)$, where N is the set of nodes and A is the set of arcs, a nonnegative weight d_{ij} is associated with each arc $(i,j) \in A$.

The shortest path problem is the problem of finding a path between the source node s and the terminal node t in a graph such that the sum of the travel times or distances on the path is minimized. In the shortest path interdiction problem, an attacker tries to maximize the shortest path sought by a defender on the network. By attacking the network with limited resources, he can partially destroy some arcs and thereby increase their weights to $d_{ij} + \gamma_{ij}$. If the arc is totally destroyed then γ_{ij} is set sufficiently large which prevents traversal of (i,j) at optimality. The attacker has a limited budget of R for interdicting some arcs in the network, each of which has an interdiction cost of c_{ij}.

The bilevel shortest path network interdiction can be formulated as follows:

$$\max_{x\in X} \min_{y} \sum_{(i,j)\in A} \left(d_{ij} + \gamma_{ij} x_{ij}\right) y_{ij}$$

$$s.t. \quad \sum_{j\in FS(i)} y_{ij} - \sum_{j\in RS(i)} y_{ij} = \begin{cases} 1 & i = s \\ -1 & i = t \\ 0 & i \in N \setminus \{s,t\} \end{cases}$$

$$y_{ij} \geq o \quad \forall\ i,j$$

where $x \in \{0,1\}^{|A|}$ is the attack vector, $y \in \{0,1\}^{|A|}$ represents the traversing vector. In addition $X = \left\{x \in \{0,1\}^{|A|} \mid c^T x \leq R\right\}$ and $FS(i) = \{ j \in N \mid (i,j) \in A\}$ $RS(i) = \{ j \in N \mid (j,i) \in A\}$ are respectively the forward star and reverse star of each node $i \in N$.

۲,۲ Stochastic shortest path network interdiction with endogenous uncertainty

In this section we introduce the notation and formulate the shortest path interdiction problema as a two-stage stochastic program. Our formulation is based on predisaster investment decision which is introduced in [۱۵]. Let p_{ij} (۰ $\leq p_{ij} \leq$ ۱) denote the probability of being traversable for link (i,j). This proabability can be decresed to q_{ij} by interdicting the link (via installing detector for example) with the cost c_{ij}. We are given a Budget B for interdicting in the links to decrease their traversable probabilities. The interdiction decision vector is denoted by $\boldsymbol{x}$=(x_{ij}), where x_{ij} is binary valued, taking the value ۱ if there is an interdiction in the link (i,j), and ۰ otherwise. After interdiction each link can be traversable or non-traversable. We use a binary-valued random variable ξ_{ij} to denote the state of link. That is, $\xi_{ij} =$ ۱, if link (i,j) is traversable, and $\xi_{ij} =$ ۰ otherwise. The vector of the random variables ξ_{ij} for all links (i,j) in E is denoted by $\boldsymbol{\xi} = (\xi_{ij})$. It represents the network realization and a specific realization of $\boldsymbol{\xi}$ is denoted by $\tilde{\boldsymbol{\xi}} = (\tilde{\xi}_{ij})$. A non-negative traversal cost t_{ij} is specified, for all (i,j). If there dose not exist a path connecting s to t in the realization, a fixed penalty cost $M < \infty$ is incurred. We assume that $M > T_{max}$ where T_{max} denoted the maximum path cost from s to t. This cost may present the cost of an alternative mode of transportation such as use of helicopter. The path vector from s to t is denoted by $\boldsymbol{x}(\tilde{\xi}) = (x_{ij}(\tilde{\xi}))$ where $x_{ij}(\tilde{\xi})$ is binary-valued, taking the value ۱ if the link (i,j) is in the shortest path of the network realization $\tilde{\xi}$ and ۰ otherwise.

The two-stage stochastic program **P** is given below.

First stage:

$$Z = \min_{\boldsymbol{x}} F(\boldsymbol{x}) = \min_{\boldsymbol{x}} E_{\xi|\boldsymbol{x}}[f(\boldsymbol{\xi})] \quad (۱)$$

$$\text{Subject to: } \textstyle\sum_{(i,j)} c_{ij}\, x_{ij} \leq B \quad (۲)$$

$$x_{ij} \in \{۰,۱\} \quad \forall\, (i,j) \in E \quad (۳)$$

Second stage:

$$z(\tilde{\xi}) = \min \sum t_{ij}\, y_{ij}(\tilde{\xi}) \quad (۴)$$

Subject to:

$$\sum_{j \in FS(i)} y_{ij}\left(\tilde{\xi}\right) - \sum_{j \in RS(i)} y_{ij}\left(\tilde{\xi}\right) = \begin{cases} 1 & i = s \\ -1 & i = t \\ 0 & i \in N \setminus \{s,t\} \end{cases} \qquad \forall i \in V \quad (۵)$$

$$y_{ij}(\tilde{\xi}) \leq \tilde{\xi}_{ij} \qquad \forall\, (i,j) \in E \quad (۶)$$

$$۰ \leq y_{ij}(\tilde{\xi}) \leq ۱ \qquad \forall\, (i,j) \in E \quad (۷)$$

where the first stage objective function, $F(\boldsymbol{x}) = E_{\xi|\boldsymbol{x}}[f(\boldsymbol{\xi})]$, is the expectation of $[f(\boldsymbol{\xi})]$, with respect to the random vector $\boldsymbol{\xi}$ for a given interdiction vector $\mathbf{y}$ and can be expand as $F(\boldsymbol{x}) = P_x(\xi = \tilde{\xi}).f(\tilde{\xi})$. Here $f(\tilde{\xi}) = z(\tilde{\xi})$, if the second stage problema for the realization $\tilde{\xi}$ is feasible, and $f(\tilde{\xi}) = M$, otherwise. Thus, its value is equal to the least traversal cost from s to t in the network realization if a path exists from s to t, or the penalti cost M, if s - t is disconnected.

Constraint (۲) is budget restriction on the total interdiction cost. Constraint set (۳) is the integrality restriction on the first stage decisión variables. The second stage objectine finction is given in (۴). Constraint set (۵) is the flow conservation constraint. Constraint set (۶) precludes flow in the links that are non-traversable in $\tilde{\xi}$. Constraint set (۷) defines the second stage flow variables which take ۰ or ۱ value but we can relax it in the interval [۰,۱] because shortest path problem is totaly unimadualar.

Note that $P_x(\xi = \tilde{\xi})$ is the probability that $\tilde{\xi}$ is realized given that the interdiction vector $\boldsymbol{x}$. Due to Independence assumption of the link failures, this can be specified as

$$P_x(\xi = \tilde{\xi}) = \Pi_{(i,j)}\{\xi_{ij}[(1 - x_{ij})p_{ij} + x_{ij}q_{ij}] + (1 - \xi_{ij})[(1 - x_{ij})(1 - p_{ij}) + x_{ij}(1 - q_{ij})]\}$$

This expression illustrate the decision-dependent nature of the probabilities in the model. Using such products directly in an optimization model seems prohibitive as they result in polynomials of degree n in x. Therefore previous approaches relied either on linearizations or convex approximations. Instead of working with these polynomials directly and trying to approximate them, a new approaches has been developed recently in [۱۶] we which proceed differently. The key observation, which will enable an efficient characterization of the resulting probability measure, is that “neighboring” measures, which can be interpreted as an application of Bayes’ rule.

For any fixed decision vector $\boldsymbol{x}$, the resulting probability measure P_x can be easily computed by applying a scaling method successively. However, to use this in an optimization model, we should be able to express this scaling not only for fixed $\boldsymbol{x}$ but as a function of $\boldsymbol{x}$. We can exploit the fact that $\boldsymbol{x}$ is binary and derive a (successive) polyhedral characterization of the resulting distribution as follows. For a given $\boldsymbol{x}$ and all $k \in \{1, \dots n\}$, let $x_k$be defined such that

$$x_i^k = \begin{Bmatrix} x_i & if\ i \leq k \\ 0 & else \end{Bmatrix}$$

and define $\pi_k^\xi = P_{x^k}(\xi)$ for all k and ξ, so π_k is an auxiliary variable carrying the probability measure induced by x^k. Then for $k < n$ the successive probability measures π_{k-1} and π_k must fulfill the set of linear inequalities:

$$\pi_k^\xi \leq \frac{q_k}{p_k}.\pi_{k-1}^\xi + 1 - x_k \qquad \forall\xi : \ \xi_k = 1$$

$$\pi_k^\xi \leq \frac{1 - q_k}{1 - p_k}.\pi_{k-1}^\xi + 1 - x_k \qquad \forall\xi : \ \xi_k = 0$$

$$\pi_k^\xi \leq \pi_{k-1}^\xi + x_k \qquad \forall\xi$$

$$\sum_\xi \pi_k^\xi = 1$$

where we naturally define $x_۰ = (۰,۰,\ldots,۰)$. Now, we can formulate the mentioned stochastic program **P**, using the decision variables x and auxiliary variables π_k introduced above. Let f be the function representing the cost in the different scenarios and E^{p_x} denote the expectation under the probability measure p_x. Then the problem **P** can be reformulated as **Ṕ**

$$\min \quad E^{p_x}(f) = \sum_{\xi} \pi_n^{\xi} . f(\xi)$$

$$\text{Subject to } \pi_k \in \mathcal{P}_k(\pi_{k-۱}, \pi_k) \qquad \forall k \in \{۱, \ldots, n\}$$

$$x \in \mathcal{X}$$

where $\mathcal{X}$ is the feasible set of problem **P**. Now, **Ṕ** is a mixed-integer linear program for minimizing the expected cost, given that the cost in each scenario is a constant for each scenario and does not depend on other decision variables. This is the case for the type of problems considered in this paper.

The above MIP has only n binary variables, but $۲^n . n$ auxiliary continuous variables and can thus become too large to be explicitly represented and solved in this form already for moderately sized problems. Solving such a NP-hard problem needs to use approximation methods or heuristic algorithms.

۳ Test problems and implementation issues

We solve the small size of such problem in a set of directed grid networks of different sizes with the same topology as the instances of [۱۱] are generated (see Fig. ۲). The instances are on a network grid with a specified number of rows and columns with randomly generated arc attributes. The characteristics of the test grids are given in Table ۱. Each node in grid position (r,c), is linked to the nodes in positions $(r+1,c)$, $(r-1,c)$, $(r,c+1)$, $(r+1,c+1)$ and $(r-1,c+1)$, all these arcs are interdictable. Generally, there are $a=(n-2)(5m-4)+5m-2$ arcs.

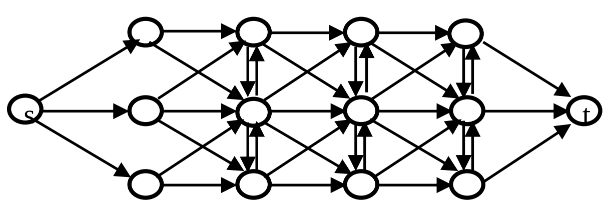

Fig ۲: Graph of a ۳*۴ test problem

The algorithms are implemented in GAMS software installed on a PC with an Intel Core i۷, ۱٫۸۰ GHz processor, and ۸ GB of RAM. The commercial solver CPLEX version ۱۱ was used to solve the MIP problems at each node of the enumeration tree.

۴ Conclusion and future works

We consider an optimization model of shortest path network interdiction problem with uncertainty. In this model, we have studied a challenging class of stochastic programs in which stochastic processes are influenced by optimization decisions which is called stochastic programs under endogenous uncertainty. In particular, we considered stochastic programs with binary decision variables that can alter the probability measures governing the random variables.

In this model, an optimal interdiction strategy is to be determined at the top level in order to interdict some network components for example by installing some detector sensors. The bottom level problem is used to find the shortest path in the surviving network after interdiction. The upper level decision maker can be police officers wants to disconnect the traversing network or increasing the cost of traversing and the bottom level decision maker can be smugglers who wants to traverse the network with the least possible cost.

As an ongoing and future direction, the decision program can be applied on a real world application with the data realization. Finding a more tractable solution algorithm can be another way to explore. Adding an extra level of protection and transformin the problem into a trilevel optimization program can be very interesting.

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