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A generalization of prime ideal

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Abstract: In this paper, we introduce the notion of pseudo-prime submodules of modules as a generalization of the prime ideal of commutative rings.
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1 Introduction

In this paper we introduce a generalization of prime ideals of a ring. We use it to define new classes of modules. We investigate some algebraic properties of these new classes.

Throughout the paper, all rings are commutative with identity and all modules are unital. For a submodule N of an R-module M, $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and *annihilator* of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(\mathbf{0} :_R M)$. If there is no ambiguity we will write $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$).

2 Pseudo-Prime Submodules

Definition 2.1. *Let M be an R-module.*

1. *A proper submodule N of M is called* pseudo-prime *if $(N :_R M)$ is a prime ideal of R.*
2. *We define the* pseudo-prime spectrum *of M to be the set of all pseudo-prime submodules of M and denote it by X_M^R. If there is no ambiguity we write only X_M instead of X_M^R. For any prime ideal $I \in X_R = \mathrm{Spec}(R)$, the collection of all pseudo-prime submodules N of M with $(N : M) = I$ is designated by $X_{M,I}$.*
3. *For a submodule N of M we define $V^M(N) = \{L \in X_M \mid L \supseteq N\}$. If there is no ambiguity we write $V(N)$ instead of $V^M(N)$.*
4. *When $X_M \neq \emptyset$, the map $\psi : X_M \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(L) = (L : M)/\mathrm{Ann}(M)$ for every $L \in X_M$, will be called* the natural map *of X_M. An R-module M is called* pseudo-primeful *if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the natural map of X_M is surjective.*
5. *M is called* pseudo-injective *if the natural map of X_M is injective.*

Remark 2.2.

1. *By our definition, the prime ideals of the ring R and pseudo-prime submodules of the R-module R are the same. This shows that pseudo-prime submodule is a generalization of the notion of prime ideal to the modules.*

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2. *We recall that a proper submodule N of an R-module M is said to be* prime *if M/N is a torsion-free $R/(N:M)$-module. The theory of prime submodules and Zariski topology on the prime spectrum of modules is studied by many algebraist (see [1, 4, 5, 6, 8, 9, 11, 12, 13]). Every prime submodule P of R-module M is pseudo-prime, because $(P:M) \in \mathrm{Spec}(R)$. However, the converse is not true in general. For example consider $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module and $N = (2,0)\mathbb{Z}$ is the submodule of M generated by $(2,0) \in M$. Then $(N:M) = (0) \in \mathrm{Spec}(\mathbb{Z})$, i.e., $N \in X_M$ though N is not a prime submodule of M. Thus in general, a pseudo-prime submodule need not be a prime submodule, i.e., $\mathrm{Spec}(M) \subsetneqq X_M$, here $\mathrm{Spec}(M)$ is the set of all prime submodules of M. This example shows that the theory of pseudo-prime submodule and the theory of prime submodule are not the same. Indeed, we can find modules such as the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$, where p is a prime integer, that has no prime submodules but every proper submodule of them is a pseudo-prime submodule. We show that the theory of pseudo-prime submodule of modules resembles to that theory of prime ideals of rings.*

Example 2.3. *Every free R-module F is pseudo-primeful (because for any prime ideal $\mathfrak{p}$ of R, $\mathfrak{p}F$ is a proper submodule of F such that $(\mathfrak{p}F : F) = \mathfrak{p}$). However, the converse is not true in general. For example consider $M = \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module. Then M is pseudo-primeful which is not free.*

Remark 2.4. *We remark that pseudo-primeful modules and primeful modules which is introduced in [10] are not the same. More precisely, every primeful module is a pseudo-primeful module. However, the converse is not true in general. For example the $\mathbb{Z}$-module $\bigoplus_p(\mathbb{Z}/p\mathbb{Z})$, where p runs over the set of all prime integers is not primeful by [10, Result 2], but it is easy to see that this is pseudo-primeful.*

Example 2.5. *We recall that an R-module M is called a* multiplication *module if every submodule N of M is of the form IM for some ideal I of R (see [3] and [7]). Every multiplication module is pseudo-injective. However, the converse is not true in general. For example consider $L = (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module, where p is a prime integer. Let Q be a pseudo-prime submodule of L. Then $(Q:L)L \subseteq Q \neq L$. Since L is a torsion $\mathbb{Z}$-module, if $(Q:L) \neq p$, then $(Q:L)L = L$ which is a contradiction. Therefore, $(Q:L) = p$. Since $L/pL \cong \mathbb{Z}/p\mathbb{Z}$, $(Q:L)L$ is a maximal submodule of L, and so $Q = (Q:L)L$. This implies that L is pseudo-injective. It is easy to check that there does not exist an ideal I of $\mathbb{Z}$ such that $(\mathbb{Z}/p\mathbb{Z}) \oplus (0) = IM$, so that M is not a multiplication module.*

We claim that every pseudo-injective finitely generated R-module is multiplication.

Lemma 2.6. *Let M be an R-module and consider the following statements.*

1. *M is a multiplication module;*
2. *M is a pseudo-injective module;*
3. *$|X_{M,\mathfrak{m}}| \leq 1$ for every maximal ideal $\mathfrak{m}$ of R;*
4. *$M/\mathfrak{m}M$ is cyclic for every maximal ideal $\mathfrak{m}$ of R.*

Then the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold. Moreover, if M is finitely generated then (4) implies (1).

In the sequel, we use the notion of pseudo-prime submodules to define another new class of modules, namely *topological module*. We present some examples of topological modules (Theorem 2.10 and Theorem 2.11) and we investigate some algebraic properties of this new class. Let Y be a subset of X_M for an R-module M. We denote the intersection of all elements in Y by $\Im(Y)$.

Definition 2.7. *Let M be an R-module.*

1. *A submodule N of M is said to be* pseudo-semiprime *if it is an intersection of pseudo-prime submodules.*

2. *A pseudo-prime submodule H of M is called* extraordinary *if $N \cap L \subseteq H$, where N and L are pseudo-semiprime submodules of M, then either $L \subseteq H$ or $N \subseteq H$.*

3. *For a submodule N of M, the* pseudo-prime radical *of N, denoted by $\mathbb{P}\mathrm{rad}(N)$, is the intersection of all pseudo-prime submodules of M containing N, that is $\mathbb{P}\mathrm{rad}(N) = \Im(V(N)) = \bigcap_{P \in V(N)} P$. If $V(N) = \emptyset$, then we put $\mathbb{P}\mathrm{rad}(N) = M$.*

4. *A submodule N of M is said to be a* pseudo-prime radical submodule *if $N = \mathbb{P}\mathrm{rad}(N)$.*

5. *M is said to be* topological *if $X_M = \emptyset$ or every pseudo-prime submodule of M is extraordinary.*

Example 2.8.

1. *Every radical ideal of a ring R is a pseudo-semiprime submodule of the R-module R. For another example, every proper submodule of a co-semisimple module is a pseudo-semiprime submodule (see [2, p. 122]).*

2. *Any prime ideal of the ring R is an extraordinary pseudo-prime submodule of the R-module R.*

3. *It is not true that every pseudo-prime submodule is extraordinary. For example consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus (\mathbb{Z}/p\mathbb{Z})$, where p is a prime integer. It is easy to see that the submodules $(0) \oplus (\mathbb{Z}/p\mathbb{Z})$, $\mathbb{Q} \oplus (0)$ and $\mathbb{Z} \oplus (0)$ of M are pseudo-prime. We deduce from $((0) \oplus (\mathbb{Z}/p\mathbb{Z})) \cap (\mathbb{Q} \oplus (0)) \subseteq \mathbb{Z} \oplus (0)$ that $\mathbb{Z} \oplus (0)$ is not extraordinary. Therefore, M is* not *a topological module.*

4. *The notion of top modules is introduced in [12] and by definition, every topological R-module is a top module. However, the converse is not true in general. For example, the above mentioned $\mathbb{Z}$-module M is a top module by [12, Example 2.6].*

5. *We recall that an R-module is called* uniserial *if its submodules are linearly ordered by inclusion (see [15]). Obviously, any uniserial module is a topological module. Hence, the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$, where p is a prime integer is a topological $\mathbb{Z}$-module.*

Theorem 2.9. *Let M be a topological R-module.*

1. *Any R-homomorphic image of M is a topological R-module.*

2. *$M_\mathfrak{p}$ is a topological $R_\mathfrak{p}$-module for each prime ideal $\mathfrak{p}$ of R.*

It is clear that if R is any ring then the pseudo-prime submodules of R (as an R-module) are the pseudo-prime ideals, and hence the R-module R is a topological module. The Theorem 2.9 shows that every cyclic R-module is a topological module. In the Theorem 2.10, we generalized this fact to multiplication modules (recall that every cyclic module is multiplication, see [3]).

Theorem 2.10. *Consider the following statements for an R-module M.*

1. *M is a multiplication module;*

2. *For every submodule N of M there exists an ideal I of R such that $V(N) = V(IM)$;*

3. *M is a topological module;*

Then the implications $(1) \Rightarrow (2) \Rightarrow (3)$ hold. Moreover, if M is finitely generated then (3) implies (1).

In the next theorem we present some examples of topological modules. For any element x of an R-module M, we define $c(x) := \bigcap\{A | A \text{ is an ideal of } R \text{ and } x \in AM\}$. We recall that an R-module M is called a *content* R-module if, for every $x \in M$, $x \in c(x)M$. Every free module, or more generally, every projective module, is a content R-module [14, p. 51]. M is a content

R-module if and only if for every family $\{A_i | i \in J\}$ of ideals of R, $(\bigcap_{i \in J} A_i)M = \bigcap_{i \in J}(A_i M)$. Also, every faithful multiplication module is a content module [7, Theorem 1.6].

Theorem 2.11. *The R-module M is topological in each of the following cases:*

1. *M is a content and pseudo-injective module.*
2. *$\mathbb{P}\mathrm{rad}(N) = \sqrt{(N : M)}M$ for each submodule N of M.*

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