46th Annual Iranian Mathematics Conference
25-28 August 2015
Yazd University
Poster A generated prefilter by a set in EQ-algebra pp.: 1–4

A prefilter generated by a set in EQ-algebras

Neda Mohtashamniya*
Departement of Mathematics, Science and Research Branch, Islamic Azad university, Kerman, Iran
Lida Torkzadeh
Departement of Mathematics, Kerman Branch, Islamic Azad university, Kerman,Iran

Abstract

In this paper we introduce the notion of a prefilter generated by a nonempty subset of an EQ-algebra E and we investigate some properties of it. After that by some theorems we characterize a generated prefilter.Then by constituting the set of all prefilters of an EQ-algebra E denoted by $PF(E)$, we show that it s an algebric lattice. Finally, we prove that, the set of all principle prefilters of an ℓEQ-algebra E is a sublattice of $PF(E)$.

Keywords: (Good, separated) EQ-algebra, ℓEQ-algebra.
Mathematics Subject Classification [2010]: 03G1, 03G05

1 Introduction

V. Novák and B. De Baets introduced a spacial algebra called EQ-algebra in [5]. An EQ-algebras have three binary (meet, multiplication and a fuzzy equality) and a top element and also a binary operation implicatin is drived from fuzzy equality. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes commutative residuated lattice. These algebras intended to develop an algebric structure of truth values for fuzzy type theory. EQ-algebras are interesting and important algebra for studing and researching and also residuated lattices [3] and BL-algebras [1, 4, 7] are particular casses of EQ-algebras.

Definition 1.1. [2] An algebra $(E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ is called an EQ-algebra where for all $a, b, c, d \in E$:

$(E1)$ $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $a \leq b$ iff $a \wedge b = a$,
$(E2)$ $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone in both arguments w.r.t. $a \leq b$,
$(E3)$ $a \sim a = 1$, (reflexivity axiom)
$(E4)$ $(a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$, (substitution axiom)
$(E5)$ $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$, (congruence axiom)
$(E6)$ $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$, (monotonicity axiom)
$(E7)$ $a \otimes b \leq a \sim b$,
for all $a, b, c \in E$.

*Speaker

The binary operations $\wedge$, $\otimes$ and $\sim$ are called meet, multiplication and fuzzy equality, respectively.

Clear, $(E, \leq)$ is a partial order. We will also put, for $a, b \in E$

$$\widetilde{a} = a \sim 1 \text{ and } a \to b = (a \wedge b) \sim a$$

The binary operation $\to$ will be called implication.

If E is a nonempty set with three binary operations $\wedge, \otimes, \sim$ such that $(E, \wedge, 1)$ is a $\wedge$-semilattice, $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone with respect to $\leq$, then $(E, \otimes, \wedge, \sim, 1)$ is an EQ-algebra, where $a \sim b = 1$, for all $a, b \in E$.

Lemma 1.2. [2] *Let $(E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. Then the following properties hold for all $a, b, c, d \in E$:*

(e_1) $a \sim b = b \sim a$,
(e_2) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$,
(e_3) $(a \to b) \otimes (b \to c) \leq (a \to c)$ *and* $(b \to c) \otimes (a \to b) \leq (a \to c)$,
(e_4) $a \sim d \leq (a \wedge b) \sim (d \wedge b)$,
(e_5) $(a \sim d) \otimes ((a \wedge b) \sim c) \leq (d \wedge b) \sim c$,
(e_6) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$,
(e_7) $a \otimes b \leq a \wedge b \leq a, b$,
(e_8) $b \leq \widetilde{b} \leq a \to b$,
(e_9) *If* $a \leq b$, *then* $a \to b = 1$, $b \to a = a \sim b$, $\widetilde{a} \leq \widetilde{b}$, $c \to a \leq c \to b$ *and* $b \to c \leq a \to c$,
(e_{10}) *If* $a \leq b \leq c$, *then* $a \sim c \leq a \sim b$ *and* $a \sim c \leq b \sim c$,
(e_{11}) $a \otimes (a \sim b) \leq \widetilde{b}$,
(e_{12}) $(a \wedge b) \to c) \otimes (d \to a) \leq (d \wedge b) \to c$.

Throughout this paper, E will be denoted an EQ-algebra unless otherwise stated.

Definition 1.3. [6] Let E be an EQ-algebra. We say that it is

(i) good, if for all $a \in E$, $\widetilde{a} = a$,
(ii) separated, if for all $a, b \in E$, $a \sim b = 1$ implies $a = b$,
(iii) semi-separated, if for all $a \in E$, $a \sim 1 = 1$ implies $a = 1$,
$iv)$ an ℓEQ-algebra, if it has a lattice reduct and for all $a, b, c, d \in E$, $((a \vee b) \sim c) \otimes (d \sim a) \leq c \sim (d \vee b)$.

Definition 1.4. [5] A nonempty subset $F \subseteq E$ is called

A prefilter of E, if for all $a, b \in E$, the following conditions hold
(PF_1) $1 \in F$,
(PF_2) $a, a \to b \in F$, then $b \in F$.

A filter of E, if F is a prefilter of E and for all $a, b, c \in E$,
(F_3) $a \to b \in F$ implies $(a \otimes c) \to (b \otimes c) \in F$.

A positive implication prefilter of E, if F is a prefilter of E and for all $a, b, c \in E$,
(IPF_4) $a \to (b \to c) \in F$ and $a \to b \in F$ imply $a \to c \in F$.

The set of all (filters)prefilters of E is denoted by $(F(E))$ $PF(E)$.

2 A generated prefilter in EQ-algebras

For a nonempty subset $X \subseteq E$, the smallest prefilter of E which contains X, i.e. $\bigcap\{F \in PF(E) : X \subseteq F\}$, is said to be a prefilter of E generated by X and will be denoted by $< X >$.

If $a \in E$ and $X = \{a\}$, we denote by $< a >$ the prefilter generated by $\{a\}$ ($< a >$ is called principal).

For $F \in PF(E)$ and $a \in E$, we denote by $F(a) =< F \cup \{a\} >$. It is clear that $a \in F$ implies $F(a) = F$.

Theorem 2.1. *Let $\emptyset \neq X \subseteq E$. Then*

$$< X >= \{a \in E : x_1 \to (x_2 \to (x_3 \to ...(x_n \to a)...)) = 1, \text{ for some } x_i \in X \text{ and } n \geq 1\}.$$

ω is the set of nonegative integers. For $a, z \in E$ and $n \in \omega$ we define $a \to^0 z = z$, $a \to^{n+1} z = a \to (a \to^n z)$. If $a = 1$, $a \to^{n+1} z$ denoted by $\widetilde{z}^{n+1}$.

Theorem 2.2. *In every EQ-algebra E, for $\emptyset \neq X \subseteq E$ we have*

$$< X >\subseteq \{a \in E : (x_1 \otimes ... \otimes x_n) \to \widetilde{a^k} = 1, \text{ for some } x_i \in X,\ n \geq 1 \text{ and } k \in \omega\}.$$

Moreover in any good EQ-algebra

$$< X >\subseteq \{a \in E : (x_1 \otimes ..., \otimes x_n) \to a = 1, \text{ for some } x_i \in X \text{ and } n \geq 1\}.$$

Theorem 2.3. *Let E be an EQ-alebra and $a, b \in E$. Then for all a, b in E the following are satisfay:*

(i) $a \leq b$ implies $< b >\subseteq< a >$,
(ii) $a^2 = a$ implies $< a >= \{z \in E : a \to \widetilde{z}^k = 1, \text{ for some } k \in \omega\}$,
(iii) If E is a good EQ-algebra and $a^2 = a$, for $a \in E$, then $< a >= \{z \in E : a \leq z\}$,
(iv) Let F be a prefilter of an ℓEQ-algebra E. Then $a \vee b \in F$ implies $F(a) \cap F(b) = F$,
(v) In an ℓEQ-algebra E, we have $< a \vee b >=< a > \cap < b >$,
(vi) $< a \wedge b >=< a > \vee < b >$,

Theorem 2.4. *(i) Let F be a prefilter of an EQ-algebra E. Then*

$$F(a) = \{z \in E : f \to (a \to^n z) = 1, \text{ for some } f \in F \text{ and } n \in \omega\}.$$

(ii) Let F be a positive implication prefilter of E. Then

$$F(a) = \{z \in E : a \to z \in F\}.$$

Let F and G be two prefilters of E. We denote $F \vee G :=< F \cup G >$.

Theorem 2.5. *Let F and $\{F_i\}_{i \in I}$ be prefilters of an ℓEQ- algebra E. Then $F \wedge (\vee_{i \in I} F_i) = \vee_{i \in I}(F \wedge F_i)$.*

A lattice L is called Brouwerian if $a \wedge (\vee_{i \in I} b_i) = \vee_{i \in I}(a \wedge b_i)$, whenever the arbitrary unions exists. Let E be a complete lattice and let a be an element of E. Then a is called compact if $a \leq \vee X$ for some $X \subseteq E$ implies that $a \leq \vee X_1$ for some $X_1 \subseteq X$. A complete lattice is called algebric if every element is the join of compact elements.
By Theorems 2.3 and 2.5 we have the following theorem.

Theorem 2.6. *Let E be an ℓEQ-algebra. Then*

(1) The lattice $(PF(E), \subseteq)$ is a complete Brouwerian lattice.

(2) If we denote by $PF_p(E)$ the family of all principal prefilter of E, then $PF_p(E)$ is a bounded sublattice of $PF(E)$.

References

[1] C.C. Chang, Algebraic analysis of many valued logics, Trans Am Math Soc 88 (1958), pp. 467-490.

[2] M. El-Zekey, V. Novák, R. Mesiar, On good EQ-algebras, Fuzzy sets and systems, vol. 178 (2011), pp. 1-23.

[3] J. A. Goguen, The logic of inexact concepts, Synthese 19 (1968-69), pp. 325-373.

[4] G. Gratzer, Lattice Theory. First Concepts and Distributive Lattices, A Series of Books in Mathematics, San Francisco : W. H. Freeman and Company (1972).

[5] L. Liu, X. Zhang, Implicative and positive implicative prefilters of EQ-algebras, Journal of Intelligent and Fuzzy Systems 26 (2014), pp. 2087-2097.

[6] V. Novák, B. De Baets, EQ-algebra, Fuzzy sets and systems, vol. 160 (2009), pp. 2956-2978.

[7] E. Turunen, Mathematics Behind Fuzzy Logic, Physica-Verlag, Heidelberg (1999).

Email: nmohtashamniya@yahoo.com
Email: ltorkzadeh@yahoo.com