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A PARALLEL METHOD FOR QR FACTORIZATION USING BLOCK HOUSEHOLDER TRANSFORMATION

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Abstract. This paper introduces a new parallel QR factorization with the modest number of processors which is based on block Householder transformation. This algorithm performs parallel QR factorization to specific blocks of smaller size than those of the initial matrix, causing an important reduction to the computations.

1. Introduction

Definition 1.1. Any real $m \times n$ matrix A can be decomposed as

$$A = QR$$

where Q is an $m \times m$ orthogonal matrix and R is an $m \times n$ upper triangular matrix. A is assumed to be full rank: rank(A)=rank(R)=n. The Householder transformation and Givens rotation can be used to compute the decomposition.

Definition 1.2. A Householder matrix is presented by

$$H = I - \beta vv^T,$$

where $\beta = 2/v^T v$ (v is a column vector). It is easy to show that H is an $m \times m$ orthogonal and symmetric matrix. The Houlseholder

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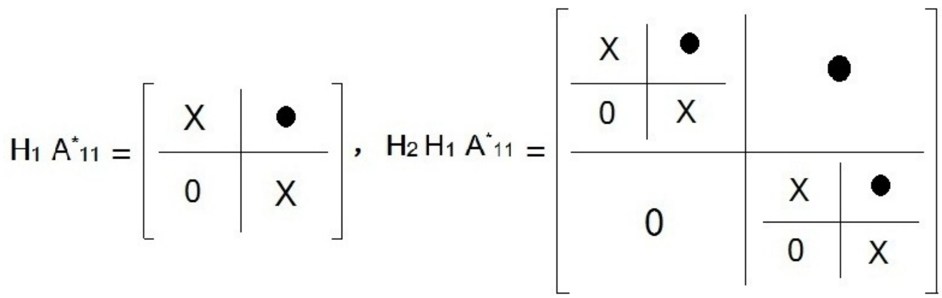

FIGURE 1. The second stage of parallel QR factorization

transformation produces large number of zeroes in one matrix operation but it can not be parallelized in a straight forward way.

2. A Parallel Method For QR Factorization

This section introduces a new parallel QR factorization algorithm using block Householder transformation introduced by Rotella and Zambettakis [1]. In this new algorithm, instead of calculating costly large problem size, a small problem size can be calculated using parallel processing similar to the method introduced by Kourniotis [2].

2.1. **Block Householder Transformation.** Let us consider a full rank matrix V and introduce the block Householder transformation

$$H(V) = I - 2V(V^TV)^{-1}V^T. \tag{2.1}$$

Theorem 2.1. *For any full column rank $m \times r$ matrix A,*

$$A = \left[A_1, A_2\right]^T$$

where A_1 is an $r \times r$ nonsingular matrix, if

$$V_A = \left[A_1 + X, A_2\right]^T,$$

with $X = P^T\sqrt{D}PA_1, \sqrt{D} = diag_{i=1}^r\sqrt{d_i}$ where the nonnegative scalar d_i and the orthogonal matrix P are defined by

$$I + (A_2A_1^{-1})^T(A_2A_1^{-1}) = P^T diag_{i=1}^r d_i P,$$

then

$$H(V_A)A = \left[-X, 0_{(m-r)\times r}\right]^T,$$

where I is an $r \times r$ identity matrix and $0_{(m-r)\times r}$ is an $(m-r) \times r$ zero matrix.

Proof. See [1]. □

2.1.1. *Application To The QR Factorization.* Let us consider a block matrix of the form

$$M = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$$

By choosing $H_1 = H(V_M)$, we have

$$H_1 M = \begin{bmatrix} -X & B_1^* \\ 0_{(m-r)\times r} & B_2^* \end{bmatrix}$$

which is an upper block triangular matrix. The Householder transformation matrix can be written

$$H_2 = \begin{bmatrix} H_2^1 & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & H_2^1 \end{bmatrix},$$

where H_2^1 and H_2^2 are the Householder transformation matrices which act, respectively, on X and B_2^*. It authorizes to calculate H_2^1 and H_2^2 in two completely independent processors.

2.2. **Description Of The New Parallel QR Algorithm.** Let A be an $m \times n$ matrix. Our aim is to find the QR factorization of A. Throughout the process, A will be partitioned into 4 blocks.

In the first stage (i=1), we separate the initial matrix $A_1(= A)$ size of $m_1 \times n_1(= m \times n)$ to blocks of sizes $m_b \times n_b$, with $m_b \geq n_b$. The upper left block is a square matrix of size $N \times N$, $N \geq \dfrac{m_1}{1+p}$, where p is the number of processors. In first step, using block Householder transformations, we obtain an upper triangular matrix. Here, if the number of rows of A_{31} is larger than N, we parallelize the multiplication of $A_{31}A_{11}^{-1}$, because this multiplication is costly. Then, in second step, we triangularize the A_{11}^* using $2^k(k = 1, 2, ..., N-1)$ processors with the following scheme (shown in figure 1).

In the second stage (i=2), the size of the matrix A_2 will be $m_2 \times n_2 = (m_1 - N) \times (n_1 - N)$, (See figure 2). In the third stage (i=3), the size of matrix A_3 will be $m_3 \times n_3 = (m_1 - 2N) \times (n_1 - 2N)$ and so on. The procedure of factorizing A ends when i reaches a value j for which it holds $n_j \leq N$.

3. Algorithmic Complexity

The complexity equations are presented only for first stage of the two steps. We know the complexity of sequential Householder transformations of an $m \times n$ matrix is $O(2n^2(m - \dfrac{n}{3}))$. The complexity of two steps and comparison of the sequential and parallel QR factorization

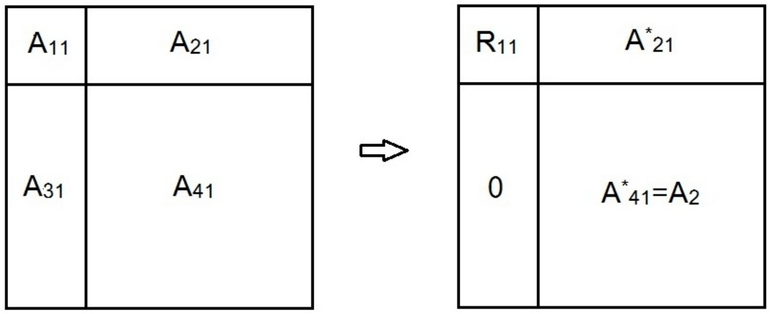

FIGURE 2. The second step scheme. Example with 2^2 processors

are shown in Table 1. At first stage, we assumed $m \geq n$ and $p \geq 2$. Hence, we can reduce the sequential complexity.

	First Step	*Second Step*	*Total*
Sequential QR factorization	$O(mN^2)$	$O((4/3)N^3)$	$O(mN^2)$
Parallel QR factorization	$O(N^3)$	$O((4/3p)N^3)$	$O(N^3)$

TABLE 1. Comparison of complexity of QR factorization in first stage.

4. Conclusion

This paper introduced a new parallel QR factorization using block Householder transformation with the modest number of processors respect to the problem size. Most of the parallel algorithms need at least $m/2$ processors. For the problems with large problem sizes, sometimes this number of processors are not available. Also, using less processors results in poor performances. The new algorithm can perform with as few as two processors. This parallel method implement QR factorization to specific blocks of smaller size than those of the initial matrix. Therefore, it can reduce computational time and numerical calculation.

References

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