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APPROXIMATE ENDPOINT GENERALIZATION FOR β-SHRINKING, β-CONVERGENT MULTIFUNCTIONS

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Abstract. In this lecture we give some generalization of approximate endpoints by using β-shrinking, β-convergent multifunctions. Also we state some applications of these results for integral equations.

1. Introduction

Let (X, d) be a metric space. Denote by $CB(X)$ the collection of all nonempty bounded and closed subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) = max\{sup_{x\in A}d(x, B), sup_{y\in B}d(y, A)\},$$

for all $A, B \in CB(X)$ where $d(x, B) = \inf_{y\in B} d(x, y)$.
Let $T : X \to 2^X$ be a multifunction. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. An element $x \in X$ is said to be an endpoint (or stationary point) of T, if $Tx = \{x\}$.

A multifunction $T : X \to 2^X$ has the approximate endpoint property, whenever $\inf_{x\in X} \sup_{y\in Tx} d(x, y) = 0$. In 2010 Amini Harandi proved the following theorem([2]).

Theorem 1.1. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ a multifunction satisfying $H(Tx, Ty) \leq \psi(d(x, y))$. for each*

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x and y in X where $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous, $\psi(t) < t$ for each $t > 0$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$. Then T has a unique endpoint if and only if T has the approximate endpoint property.

Immediately, Moradi and Khojasteh[3] defined the concept of generalized weak contractive multifunction as following:

The multifunction T is said to be generalized weak contractive, if there exists an upper semicontinuous mapping $\psi : [0, \infty) \to [0, \infty)$ with $\psi(t) < t$ for all $t > 0$, such that $H(Tx, Ty) \le \psi(N(x, y))$ for all $x, y \in X$ where

$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

They proved the following theorem which is a generalization of Theorem 1.1.

Theorem 1.2. [3] *Let (X,d) be a complete metric space and $T : X \to CB(X)$ a generalized weak contractive multifunction where $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous mapping with $\psi(t) < t$ for all $t > 0$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$. Then T has a unique endpoint if and only if T has the approximate endpoint property.*

Now, we define some new concepts and give some generalizations for endpoint of multifunctions.
We say that the function $\psi : [0, +\infty) \to [0, +\infty)$ is upper semicontinuous whenever $\limsup_{\lambda\to\lambda_0} \psi(\lambda) \le \psi(\lambda_0)$.
Let (X, d) be a metric space. The multifunction T is said to be β-generalized weak contractive if there exist the functions $\beta : 2^X \times 2^X \to [0, \infty)$ and $\psi : [0, +\infty) \to [0, +\infty)$ such that $\beta(Tx, Ty)H(Tx, Ty) \le \psi(N(x, y))$ for all $x, y \in X$ where ψ is an upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$.

Let (X, d) be a metric space, $T : X \to 2^X$ a multifunction and $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping. We say that T is β-shrinking whenever if $\{x_n\}$ be a sequence in X such that $\lim_{n\longrightarrow\infty} diam(Tx_n) = 0$, then there exists $N \in \mathbb{N}$ such that $\beta(Tx_n, Tx_m) \ge 1$ for all $m > n \ge N$.
A multifunction T is said to be β-convergent whenever for each sequence $\{x_n\}$ with $x_n \to x$, then there exists $N \in \mathbb{N}$ such that $\beta(Tx_n, Tx) \ge 1$ for all $n \ge N$.

2. Main results

Now, we are ready to state and prove our main results.

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ a β-shrinking and β-convergent multifunction satisfying*

$$\beta(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y))$$

for all x, y in X where ψ is an upper semicontinuous function with $\psi(t) < t$ for all $t > 0$. Then T has an endpoint if and only if T has the approximate endpoint property.

Now, we define the condition(G) as following:
(G) If $A, B \subset X$ such that $A \nsubseteq B$ or $B \nsubseteq A$, then $\beta(A, B) \geq 1$.

Corollary 2.2. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ a β-shrinking and β-convergent multifunction satisfying condition (G) and*

$$\beta(Tx, Ty)H(Tx, Ty) \leq kd(x, y)$$

for all x, y in X where $k \in [0, 1)$. If T has the approximate endpoint property, then there exists $x_0 \in X$ such that $Fix(T) = End(T) = \{x_0\}$.

The following theorem generalizes theorem 1.2.

Theorem 2.3. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ a β-shrinking, β-convergent and β-generalized weak contractive multifunction Then T has an endpoint if and only if T has the approximate endpoint property.*

Application. Now, we give an application of the last results. We use these results to find solution for integral equations. We used a method that is similar to the results of [4].
Consider the following integral equation:

$$u(t) = \int_0^L K(t, s, u(s))ds + g(t) \tag{2.1}$$

for all $t \in I$, where $I = [0, L]$ and $L > 0$. Also by $C(I)$, denote the set of all continuous functions $u : I \to \mathbb{R}$. One can easily see that $(C(I), d)$ is a complete metric space with the metric defined by:

$$d(u, v) = \sup_{t \in I} | u(t) - v(t) | .$$

As we know, we can consider a selfmap $T : X \to X$ with $Tx = y$, as the multifunction $T : X \to 2^X$ defined by $Tx = \{y\}$. In this case we have $H(Tx, Ty) = d(Tx, Ty)$ for all $x, y \in X$.
Now we are ready to state the following theorem.

Theorem 2.4. *Suppose that the following statements hold:*
(i) *The functions* $K : I \times I \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous.*
(ii) *There exists a continuous function* $G : I \times I \to \mathbb{R}$ *such that*

$$| K(t,s,x) - K(t,s,y) | \leq G(t,s)\frac{|x-y|}{2}$$

for all $x, y \in \mathbb{R}$ *and* $t, s \in I$.
(iii) $\sup_{t\in I} \int_0^L G^2(t,s)ds \leq \frac{1}{L}$.
(iv) $\inf_{u\in C(I)} \sup_{t\in I} |u(t) - \int_0^L K(t,s,u(s))ds - g(t)| = 0$.
Then the integral equation 2.1 has a solution.

Theorem 2.5. *Suppose that the following statements hold:*
(i) *The functions* $K : I \times I \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous.*
(ii) *There exists a function* $\beta : 2^{C(I)} \times 2^{C(I)} \to \mathbb{R}$ *and* $\alpha : 2^{\mathbb{R}} \times 2^{\mathbb{R}} \to \mathbb{R}$ *such that for all* $u, v \in C(I)$ *we have*

$$\beta(u,v) = \sup_{t\in I} \alpha(u(t), v(t)).$$

Also $\beta(A,B) \geq 1$ *whenever* A *or* B *is a singleton subsets of* $C(I)$.
(iii) *There exists a continuous function* $G : I \times I \to \mathbb{R}$ *such that*

$$| K(t,s,x) - K(t,s,y) | \leq G(t,s)\frac{J(x,y)}{2}$$

for all $x, y \in \mathbb{R}$ *and* $t, s \in I$.
(iv) $\sup_{t\in I} \alpha^2(\int_0^L K(t,s,u(s))ds + g(t), \int_0^L K(t,s,v(s))ds + g(t))$.
$\int_0^L G^2(t,s)ds \leq \frac{1}{L}$.
(v) $\inf_{u\in C(I)} \sup_{t\in I} |u(t) - \int_0^L K(t,s,u(s))ds - g(t)| = 0$.
Then the integral equation 2.1 has a solution.

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