



The Extended Abstracts of
the 44th Annual Iranian Mathematics Conference
27-30 August 2013, Ferdowsi University of Mashhad, Iran.

ON NEAR ARMENDARIZ IDEALS

KHADIJEH KHALILNEZHAD KIASARI¹ AND HAMID HAJ
SEYYED JAVADI²

^{1,2}*Department Of Mathematics, Shahed University, P. O. Box: 18155/159,
Tehran, Iran.*

¹*kh.khalilnezhad@shahed.ac.ir; ²h.s.javadi@shahed.ac.ir*

ABSTRACT. In this paper, we introduce the concept of near Armendariz ideals and record some results involving them.

1. INTRODUCTION

Throughout this paper, all rings are associative with identity. In [1], M. B. Rege et al. introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^n b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for all i and j . (The converse is always true). The term of an Armendariz ring was chosen because E. Armendariz [2, lemma 1] had noted that a reduced ring satisfies this condition. In [3], Sh. Ghalandarzadeh et al. introduced the notion of an Armendariz and abelian ideals. A left ideal I of R is called Armendariz if whenever polynomials $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^n b_j x^j \in R[x]$ satisfy $f(x)g(x) \in r_{R[x]}(I[x])$ we have $a_i b_j \in r_R(I)$ for all i and j . Also a left ideal I of R is called abelian if for each idempotent element $e \in R$, $er - re \in r_R(I)$ for any $r \in R$. Over a reduced ring R , G. F. Birkenmeier, [4, lemma 3.4], proved that $g_j f_i = 0$ for each $1 \leq i \leq m$, $1 \leq j \leq n$ and $(fo)g = 0$ whenever $(x)fo(x)g = 0$ where $(x)f = \sum_{i=0}^m f_i x^i$, $(x)g = \sum_{j=0}^n g_j x^j \in R[x]$. Due to Ghalandarzadeh et al. [5],

2010 *Mathematics Subject Classification.* Primary 16D25; Secondary 16N99, 16S50.

Key words and phrases. Near Armendariz ideal; abelian ideal; IFP ideal.

such rings (possibly not reduced) that satisfy Birkenmeier's result, are called near Armendariz. The binary operation of substitution, denoted by o , of one polynomial into another is both natural and important in the theory of polynomials. We adopt the convention that for $(x)f$, $(x)g \in R[x]$, with $(x)g = \sum_{j=0}^n g_j x^j$, $(x)fo(x)g = \sum_{j=0}^n g_j ((x)f)^j$; However, the operation " o " left distributes but does not right distribute over addition. Thus $(R[x], +, o)$ forms a left nearring but not a ring. Henceforth, unless indicated otherwise, $R[x]$ denotes the nearring of polynomials $(R[x], +, o)$ and $R_0[x]$ the subnearring of polynomials with zero constant term. In this paper we study near Armendariz ideals; this concept is related to that of near Armendariz rings.

2. MAIN RESULTS

In this section we define and study near Armendariz (one-sided) ideals. All our left-sided concepts and results have right-sided counterparts. The right annihilator of a subset A of a ring R is denoted by $r_R(A)$ or $r(A)$ (when R is clear from the context). We begin with the following definition.

Definition 2.1. Let R be a ring.

A left ideal I of R is called near Armendariz if whenever polynomials $(x)f = \sum_{i=0}^m f_i x^i$, $(x)g = \sum_{j=0}^n g_j x^j \in R[x]$ satisfy $(x)fo(x)g \in r_{R[x]}(I[x])$ then $g_j f_i \in r_R(I)$ for each $1 \leq i \leq m$, $1 \leq j \leq n$ and $(f_0)g \in r_R(I)$.

Proposition 2.2. Let R be a ring and I be a near Armendariz left ideal of R . If $(x)f_1, (x)f_2, \dots, (x)f_n \in R[x]$ are such that $(x)f_1 o \dots o (x)f_n \in r_{R[x]}(I[x])$, then $a_n a_{n-1} \dots a_1 \in r_R(I)$ where a_i is a coefficient of f_i .

Proposition 2.3. If I is a near Armendariz left ideal of R , then I is an abelian left ideal.

The following example shows that the converse of proposition 2.3, does not hold.

Example 2.4. Let S be an abelian ring and

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in S \right\}.$$

Notice that R has trivial idempotent. Thus R is an abelian ring. Next let

$$I = \begin{pmatrix} 0 & S & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then I is an abelian left ideal, because $er - re \in r_R(I)$. Consider

$$(x)f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} x, \quad (x)g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x^2$$

in $R[x]$. Then $(x)fo(x)g \in r_{R[x]}(I[x])$, but

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \notin r_R(I).$$

So I is not a near Armendariz left ideal of R .

Proposition 2.5. *Let R be a near Armendariz ring and I be an ideal of R . Then $R/r_R(I)$ is a near Armendariz ring.*

Theorem 2.6. *If R is a near Armendariz ring, then each left ideal of R is a near Armendariz left ideal.*

Proof. Let R be a near Armendariz ring and I be a left ideal of R . Suppose that $(x)f = \sum_{i=0}^m f_i x^i$, $(x)g = \sum_{j=0}^n g_j x^j$ are elements of $R[x]$ such that $(x)fo(x)g \in r_{R[x]}(I[x])$. Next consider $\bar{R} = R/r_R(I)$. Let $(x)\bar{f} = \bar{f}_0 + \bar{f}_1 x + \dots + \bar{f}_n x^n$, $(x)\bar{g} = \bar{g}_0 + \bar{g}_1 x + \dots + \bar{g}_n x^n \in \bar{R}[x]$ such that $(x)\bar{f}o(x)\bar{g} = 0$. Thus, $0 = (x)\bar{f}o(x)\bar{g} = (\bar{g}_0 + \bar{g}_1 \bar{f}_0 + \dots + \bar{g}_n \bar{f}_0) + \sum_{p=1}^{n^2} \left(\sum_{j=\lfloor \frac{p}{n} \rfloor}^n \bar{g}_j \bar{c}_p^{(j)} \right) x^p$ where $\bar{c}_p^{(j)} = \sum_{u_1+\dots+u_j=p} \bar{f}_{u_1} \bar{f}_{u_2} \dots \bar{f}_{u_j}$ for $p \in \{1, 2, \dots, n^2\}$. Hence $(\bar{f}_0)\bar{g} = \bar{g}_n \bar{f}_0^n + \dots + \bar{g}_1 \bar{f}_0 + \bar{g}_0 = 0$. It follows $\left(\sum_{j=\lfloor \frac{p}{n} \rfloor}^n g_j c_p^{(j)} \right) \in r_R(I)$ and so $\left(\sum_{j=\lfloor \frac{p}{n} \rfloor}^n v g_j c_p^{(j)} \right) = 0$ for all $v \in I$ by proposition 2.5. This means that $(x)fov(x)g = 0$ in $R[x]$. Since R is a near Armendariz ring, for $1 \leq i, j \leq n$, $vg_j f_i = 0$ for all $v \in I$. Then $g_j f_i \in r_R(I)$, for $1 \leq i, j \leq n$. Also from hypothesis $(x)\bar{f}o(x)\bar{g} = 0$, it follows $(f_0)g \in r_R(I)$. \square

Let R be a ring. The trivial extension of R is defined to be the ring $T(R, R) = R \oplus R$ with the usual addition and the multiplication $(r_1, r_2)(r'_1, r'_2) = (r_1 r'_1, r_1 r'_2 + r_2 r'_1)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & r' \\ 0 & r \end{pmatrix}$, where $r, r' \in R$. Next we give an example of a nonzero near Armendariz left ideal of a non-near Armendariz ring.

Example 2.7. Let $S = \mathbb{Z}_8$ and $R = T(S, S)$. Then R is not near Armendariz because $(x)f = (\bar{4}, \bar{2})x$, $(x)g = (\bar{2}, 0)x^2 \in R[x]$. Then $(x)fo(x)g = 0$, but $(\bar{2}, 0)(\bar{4}, \bar{2}) \neq 0$. Write $a = \begin{pmatrix} 0 & \bar{2} \\ 0 & 0 \end{pmatrix}$, $I = Ra$ and $r(I) = r_R(I)$, then

$$r(I) = \left\{ \begin{pmatrix} r & b \\ 0 & r \end{pmatrix} \mid r \in \{0, \bar{2}\}, b \in \mathbb{Z}_8 \right\}.$$

Since $r(I)$ is an ideal of R and $R/r(I)$ is a reduced ring, I is a near Armendariz left ideal of R .

Recall that a one-sided ideal I of a ring R has the insertion of factors property (or simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$.

By the following example, we show that I has the IFP, but I is not near Armendariz ideal.

For a ring R , we denote by $T_n(R)$ the n -by- n upper triangular matrix ring over R .

Example 2.8. Let D be a domain, $R = T_2(D)$ and $I = \begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix}$. Let

$$0 \neq \alpha = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, 0 \neq \beta = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R.$$

It is easily shown that $\alpha\beta \in I$ if and only if $c = 0$. Thus if $c = 0$ then $\alpha R\beta \subseteq I$. This implies that I has the IFP. Next consider $(x)f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}x$, $(x)g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}x + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}x^2 \in R[x]$ such that $(x)fo(x)g \in r_{R[x]}(I[x])$, but $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin r_R(I)$. Thus I is not a near Armendariz left ideal.

Proposition 2.9. *Let R be a ring and let I be a near Armendariz left ideal of R , then $r_R(I)$ has the IFP.*

REFERENCES

1. M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. 73 (1997), no. 1, 14-17.
2. E. P. Armendariz, *A note on extensions of Baer and P.P.-rings*, J. Aust. Math. Soc. 18 (1974), 470-473.
3. Sh. Ghalandarzadeh, H. Haj Seyyed Javadi, M. Khoramdel and M. Shamsaddini Fard, *On Armendariz ideals*, Bull. Korean Math. Soc. 47 (2010), no. 5, 883-888
4. G. F. Birkenmeier and F. K. Huang, *Annihilator Conditions on polynomials*, Comm. Algebra 29 (2001), 2097-2112.
5. Sh. Ghalandarzadeh and P. Malakooti Rad, *On π -near Armendariz rings*, Asian-European J. Math. 2 (2009), no. 1, 77-83.