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# ON NEAR ARMENDARIZ IDEALS

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Abstract. In this paper, we introduce the concept of near Armendariz ideals and record some results involving them.

## 1. Introduction

Throughout this paper, all rings are associative with identity. In [1], M. B. Rege et al. introduced the notion of an Armendariz ring. They defined a ring $R$ to be an Armendariz ring if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for all $i$ and $j$. (The converse is always true). The term of an Armendariz ring was chosen because $E$. Armendariz [2, lemma 1] had noted that a reduced ring satisfies this condition. In [3], Sh. Ghalandarzadeh et al. introduced the notion of an Armendariz and abelian ideals. A left ideal $I$ of $R$ is called Armendariz if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) \in r_{R[x]}(I[x])$ we have $a_i b_j \in r_R(I)$ for all $i$ and $j$. Also a left ideal $I$ of $R$ is called abelian if for each idempotent element $e \in R, er - re \in r_R(I)$ for any $r \in R$. Over a reduced ring $R$, G. F. Birkenmeier, [4, lemma 3.4], proved that $g_j f_i = 0$ for each $1 \leq i \leq m$, $1 \leq j \leq n$ and $(fo)g = 0$ whenever $(x)fo(x)g = 0$ where $(x)f = \sum_{i=0}^{m} f_i x^i$, $(x)g = \sum_{j=0}^{n} g_j x^j \in R[x]$. Due to Ghalandarzadeh et al. [5],

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such rings (possibly not reduced) that satisfy Birkenmeier's result, are called near Armendariz. The binary operation of substitution, denoted by $o$, of one polynomial into another is both natural and important in the theory of polynomials. We adopt the convention that for $(x)f$, $(x)g \in R[x]$, with $(x)g = \sum_{j=0}^{n} g_j x^j$, $(x)fo(x)g = \sum_{j=0}^{n} g_j((x)f)^j$; However, the operation "$o$" left distributes but does not right distribute over addition. Thus $(\mathrm{R[x]}, +, o)$ forms a left nearring but not a ring. Henceforth, unless indicated otherwise, $R[x]$ denotes the nearring of polynomials $(\mathrm{R[x]}, +, o)$ and $R_0[x]$ the subnearring of polynomials with zero constant term. In this paper we study near Armendariz ideals; this concept is related to that of near Armendariz rings.

## 2. Main results

In this section we define and study near Armendariz (one-sided) ideals. All our left-sided concepts and results have right-sided counterparts. The right annihilator of a subset $A$ of a ring $R$ is denoted by $r_R(A)$ or $r(A)$ (when $R$ is clear from the context). We begin with the following definition.

**Definition 2.1.** Let R be a ring.
A left ideal $I$ of $R$ is called near Armendariz if whenever polynomials $(x)f = \sum_{i=0}^{m} f_i x^i$, $(x)g = \sum_{j=0}^{n} g_j x^j \in R[x]$ satisfy $(x)fo(x)g \in r_{R[x]}(I[x])$ then $g_j f_i \in r_R(I)$ for each $1 \leq i \leq m$, $1 \leq j \leq n$ and $(f_0)g \in r_R(I)$.

**Proposition 2.2.** *Let $R$ be a ring and $I$ be a near Armendariz left ideal of $R$. If $(x)f_1, (x)f_2, \ldots, (x)f_n \in R[x]$ are such that $(x)f_1 o \ldots o(x)f_n \in r_{R[x]}(I[x])$, then $a_n a_{n-1} \ldots a_1 \in r_R(I)$ where $a_i$ is a coefficient of $f_i$.*

**Proposition 2.3.** *If $I$ is a near Armendariz left ideal of $R$, then $I$ is an abelian left ideal.*

The following example shows that the converse of proposition 2.3, does not hold.

**Example 2.4.** Let $S$ be an abelian ring and

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in S \right\}.$$



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Notice that $R$ has trivial idempotent.Thus $R$ is an abelian ring. Next let

$$I = \begin{pmatrix} 0 & S & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $I$ is an abelian left ideal, because $er - re \in r_R(I)$. Consider

$$(x)f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} x, \quad (x)g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x^2$$

in $R[x]$. Then $(x)fo(x)g \in r_{R[x]}(I[x])$, but

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \notin r_R(I).$$

So $I$ is not a near Armendariz left ideal of R.

**Proposition 2.5.** *Let $R$ be a near Armendariz ring and $I$ be an ideal of $R$. Then $R/r_R(I)$ is a near Armendariz ring.*

**Theorem 2.6.** *If $R$ is a near Armendariz ring, then each left ideal of $R$ is a near Armendariz left ideal.*

*Proof.* Let $R$ be a near Armendariz ring and $I$ be a left ideal of $R$. Suppose that $(x)f = \sum_{i=0}^{m} f_i x^i$, $(x)g = \sum_{j=0}^{n} g_j x^j$ are elements of $R[x]$ such that $(x)fo(x)g \in r_{R[x]}(I[x])$. Next consider $\bar{R} = R/r_R(I)$. Let $(x)\bar{f} = \bar{f}_0 + \bar{f}_1 x + \ldots + \bar{f}_n x^n$, $(x)\bar{g} = \bar{g}_0 + \bar{g}_1 x + \ldots + \bar{g}_n x^n \in \bar{R}[x]$ such that $(x)\bar{f}o(x)\bar{g} = 0$. Thus, $0 = (x)\bar{f}o(x)\bar{g} = (\bar{g}_0 + \bar{g}_1 \bar{f}_0 + \ldots + \bar{g}_n \bar{f}_0^n) + \sum_{p=1}^{n^2} \left( \sum_{j=\left[\frac{p}{n}\right]}^{n} \bar{g}_j \bar{c}_p^{(j)} \right) x^p$ where $\bar{c}_p^{(j)} = \sum_{u_1+\ldots+u_j=p} \bar{f}_{u_1} \bar{f}_{u_2} \ldots \bar{f}_{u_j}$ for $p \in \{1, 2, \ldots, n^2\}$. Hence $(\bar{f}_0)\bar{g} = \bar{g}_n \bar{f}_0^n + \ldots + \bar{g}_1 \bar{f}_0 + \bar{g}_0 = 0$. It follows $\left( \sum_{j=\left[\frac{p}{n}\right]}^{n} g_j c_p^{(j)} \right) \in r_R(I)$ and so $\left( \sum_{j=\left[\frac{p}{n}\right]}^{n} v g_j c_p^{(j)} \right) = 0$ for all $v \in I$ by proposition 2.5. This means that $(x)fov(x)g = 0$ in $R[x]$. Since $R$ is a near Armendariz ring, for $1 \leq i, j \leq n$, $v g_j f_i = 0$ for all $v \in I$. Then $g_j f_i \in r_R(I)$, for $1 \leq i, j \leq n$. Also from hypothesis $(x)\bar{f}o(x)\bar{g} = 0$, it follows $(f_0)g \in r_R(I)$. □

Let $R$ be a ring. The trivial extension of $R$ is defined to be the ring $T(R, R) = R \oplus R$ with the usual addition and the multiplication $(r_1, r_2)(r'_1, r'_2) = (r_1 r'_1, r_1 r'_2 + r_2 r'_1)$. This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & r' \\ 0 & r \end{smallmatrix}\right)$, where $r, r' \in R$. Next we give an example of a nonzero near Armendariz left ideal of a non-near Armendariz ring.



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**Example 2.7.** Let $S = \mathbb{Z}_8$ and $R = T(S,S)$. Then $R$ is not near Armendariz because $(x)f = (\overline{4},\overline{2})x$, $(x)g = (\overline{2},0)x^2 \in R[x]$. Then $(x)fo(x)g = 0$, but $(\overline{2},0)(\overline{4},\overline{2}) \neq 0$. Write $a = \begin{pmatrix} 0 & \overline{2} \\ 0 & 0 \end{pmatrix}$, $I = Ra$ and $r(I) = r_R(I)$, then

$$r(I) = \left\{ \begin{pmatrix} r & b \\ 0 & r \end{pmatrix} \mid r \in \{0,\overline{2}\}, b \in \mathbb{Z}_8 \right\}.$$

Since $r(I)$ is an ideal of $R$ and $R/r(I)$ is a reduced ring, $I$ is a near Armendariz left ideal of $R$.

Recall that a one-sided ideal $I$ of a ring $R$ has the insertion of factors property (or simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a,b \in R$.

By the following example, we show that $I$ has the IFP, but $I$ is not near Armendariz ideal.
For a ring $R$, we denote by $T_n(R)$ the n-by-n upper triangular matrix ring over $R$.

**Example 2.8.** Let $D$ be a domain, $R = T_2(D)$ and $I = \begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix}$. Let

$$0 \neq \alpha = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, 0 \neq \beta = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R.$$

It is easily shown that $\alpha\beta \in I$ if and only if $c = 0$. Thus if $c = 0$ then $\alpha R \beta \subseteq I$. This implies that $I$ has the IFP. Next consider $(x)f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x, (x)g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x^2 \in R[x]$ such that $(x)fo(x)g \in r_{R[x]}(I[x])$, but $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin r_R(I)$. Thus $I$ is not a near Armendariz left ideal.

**Proposition 2.9.** *Let $R$ be a ring and let $I$ be a near Armendariz left ideal of $R$, then $r_R(I)$ has the IFP.*

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