A comparison between finite-difference schemes used for 2D acoustic wave modeling

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ABSTRACT
In this paper we investigate numerical characteristics of different finite-difference schemes used for 2D acoustic wave propagation simulation. Numerical dispersion is analyzed as a function of temporal frequency to demonstrate directional propagation dependency of different schemes. 5-point, 7-point and 9-point explicit schemes are compared using metrics relevant to acoustic wave simulation. It is shown that the 7-point hexagonal scheme has better isotropic characteristics than conventional 5-point and 9-point rectangular schemes.

Keywords: Finite difference, hexagonal grids, rectangular grids, Numerical dispersion

INTRODUCTION

Wave extrapolation is crucial for seismic modeling, imaging (reverse-time migration), and full waveform inversion. The most popular and straightforward way to implement wave extrapolation is the method of explicit finite differences (FD). Despite explicit approaches popularity they suffers from numerical dispersion and stability problems (Wu et al, 1996). The basic idea behind FD methods is approximating the partial derivatives by Taylor series expansions near the point of interest (Smith, 1985). Conventionally rectangular FD schemes are used to approximate multidimensional spatial partial differential operator but they are not the most efficient schemes. Sampling 2D isotropic functions on hexagonal grid is significantly more efficient than sampling on rectangular grid (Dudgeon et al, 1984).
In this paper we review the 5-point, 7-point and 9-point grid FD schemes then we make a brief discussion about stability condition and finally analyze and compare numerical dispersion of schemes.

5-point, 7-point and 9-point FD schemes

The following acoustic-wave equation is widely used in seismic modeling and reverse-time migration:

$$\frac{\partial^2 P}{\partial t^2} = c^2 \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2}\right) \tag{1}$$

Where P is the pressure field and *c* is the wave propagation velocity. By using 2D Taylor series expansion we can discretize acoustic wave equation with second order approximations in both time and space. For 5-point rectangular grid the discretized acoustic wave equation will be:

$$P_{i,j}^{n+1} = \lambda^2 (P_{i+1,j}^{n} + P_{i,j+1}^{n} + P_{i-1,j}^{n} + P_{i,j-1}^{n}) + (2-4\lambda^2) P_{i,j}^{n} - P_{i,j}^{n-1} + O(\Delta t^2, \Delta x^2) \tag{2}$$

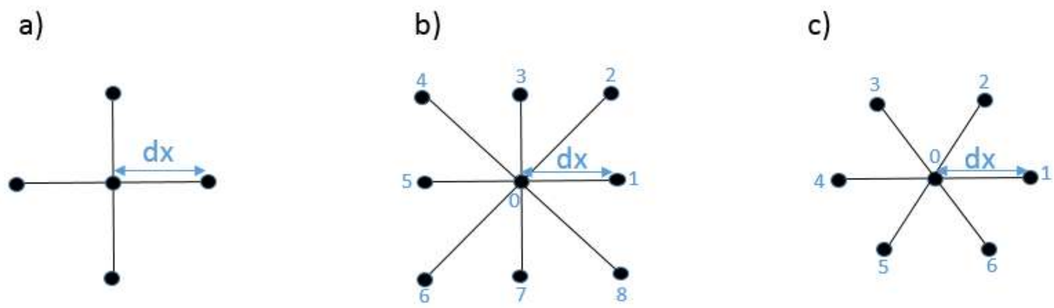

Figure 1. a) 5-point grid b) 9-point grid c) 7-point grid.

The 7-point hexagonal one is as following (Fabero, 2001):

$$P_0^{n+1} = \frac{2\lambda^2}{3}(P_1^n + P_2^n + P_3^n + P_4^n + P_5^n + P_6^n) + (2-4\lambda^2)P_0^n - P_0^{n-1} + O(\Delta t^2, \Delta x^2) \quad (3)$$

Also 9-point rectangular grid will be:

$$P_0^{n+1} = \lambda^2\alpha(P_1^n + P_3^n + P_5^n + P_7^n) + \frac{\lambda^2(1-\alpha)}{2}(P_2^n + P_4^n + P_6^n + P_8^n)$$
$$+2(1-\lambda^2(1+\alpha))P_0^n - P_0^{n-1} + O(\Delta t^2, \Delta x^2) \quad (4)$$

Where $\alpha = 1/2$ or $\alpha = 2/3$ in acoustic wave simulation (Bilbao, 2013). In all above equations, $\lambda = \frac{c\Delta t}{\Delta x}$ and *n* is time step.

Stability Consideration

The stability condition for the hexagonal case requires more care because the region to be considered is a hexagon. Consequently, many studies have given the wrong stability limits $\lambda \le \sqrt{\frac{1}{2}}$ but the correct stability limit for hexagonal grid is $\lambda^2 \le \frac{2}{3}$ (Fabero, 2001).

Dispersion Analysis

The derivation of the numerical dispersion relation is presented in this section. To derive the ratio of numerical phase speed to true phase speed, a monochromatic wave expression is substituted into the corresponding schemes as a test solution,

$$P = P_0 e^{j(k_x x + k_z z - \omega t)} \quad (5)$$

Where k_x and k_z are wave number components in x and z directions respectively, and ω is angular frequency. So the dispersion error equation for each scheme is derived. Phase speed error for 5-point scheme will be:

$$\frac{\hat{c}}{c} = \frac{\omega}{ck} = \frac{2\sin^{-1}\left(\lambda\sqrt{\sin^2\left(\frac{k_x\Delta x}{2}\right) + \sin^2\left(\frac{k_z\Delta x}{2}\right)}\right)}{\lambda\sqrt{(k_x\Delta x)^2 + (k_z\Delta x)^2}} \quad (6)$$

Phase speed error for 9-point scheme:

$$\frac{\hat{c}}{c} = \frac{\omega}{ck} = \frac{2\sin^{-1}\left(\lambda\sqrt{\sin^2\left(\frac{k_x\Delta x}{2}\right) + \sin^2\left(\frac{k_z\Delta x}{2}\right) - 2(1-\alpha)\sin^2\left(\frac{k_x\Delta x}{2}\right)\sin^2\left(\frac{k_z\Delta x}{2}\right)}\right)}{\lambda\sqrt{(k_x\Delta x)^2 + (k_z\Delta x)^2}} \quad (7)$$

And phase speed error for 7-point scheme:

$$\frac{\hat{c}}{c} = \frac{\omega}{ck} = \frac{2\sin^{-1}\left(\sqrt{\frac{2}{3}}\lambda\sqrt{\sin^2\left(\frac{k_x\Delta x}{2}\right) + 2\sin^2\left(\frac{k_x\Delta x}{4}\right) + 2\sin^2\left(\frac{\sqrt{3}k_z\Delta x}{4}\right) - 4\sin^2\left(\frac{k_x\Delta x}{4}\right)\sin^2\left(\frac{\sqrt{3}k_z\Delta x}{4}\right)}\right)}{\lambda\sqrt{(k_x\Delta x)^2 + (k_z\Delta x)^2}} \quad (8)$$

As a standard routine for comparison of schemes we compare them in their maximum stability limits (Walstijn, 2008).

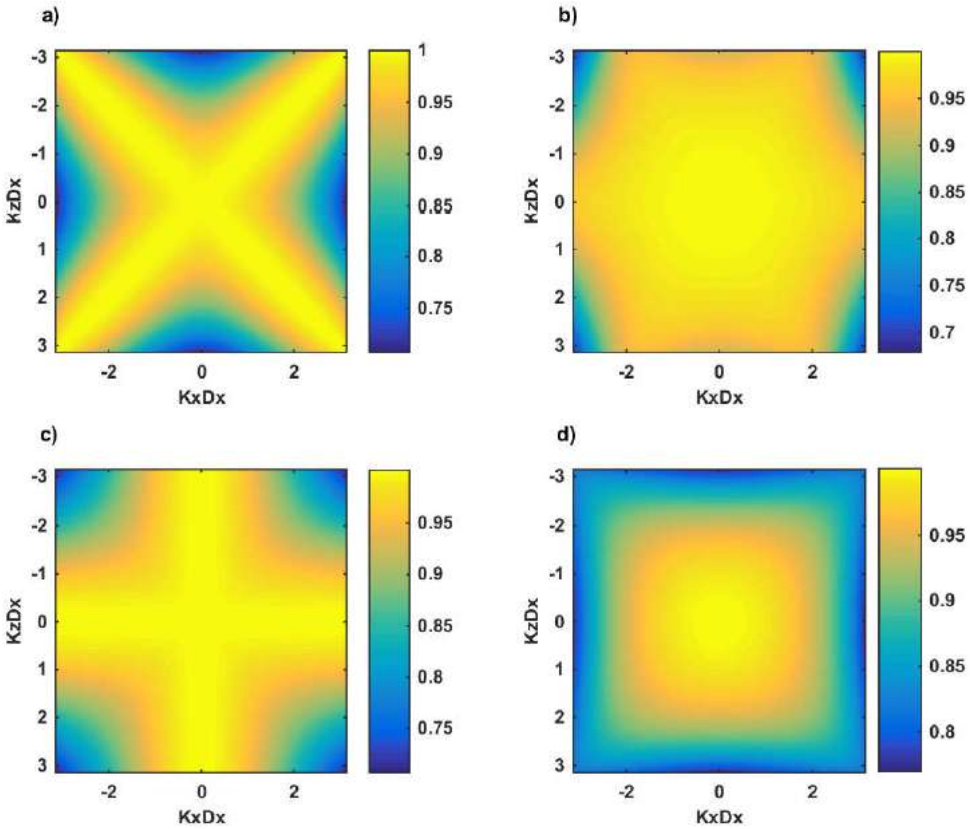

Figure 2. Relative phase velocity error for four schemes at their maximum stability limit. a) 5-point scheme. b) 7-point scheme. c) 9-point scheme with $\alpha=1/2$. d) 9-point scheme with $\alpha=2/3$.

According to Fig. 2 the 7-point scheme is more isotropic and has less dispersion error in a wide frequency band. To test our comparison we did a numerical simulation for 9-point scheme with $\alpha=2/3$ and 7-point scheme also. The model is a homogeneous case with wave speed *c*=1500 m/s. A Ricker wavelet with a dominant frequency of 50 Hz is located at the center of the model as the source. For numerical simulations, the grid size is $\Delta x = 1$ m and $\Delta t = 0.5$ ms.

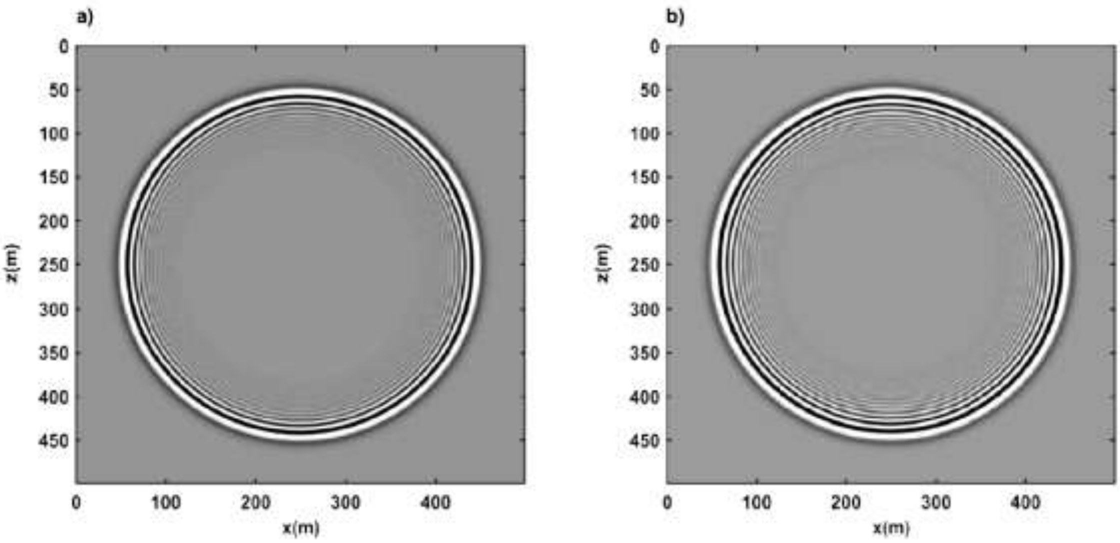

Figure 3. Pressure wave-field generated by a) 7-point scheme and b) 9-point scheme.

Simulation result clearly shows that the dispersion error in 7-point scheme is less than 9-point scheme.

CONCLUSION

Traditionally the FD schemes are derived for rectangular grids (5-point and 9-point). We compared them with a 7-point FD scheme that is based on the hexagonal geometrical grid. The main problem in using explicit FD operators is dispersion error and the 7-point operator possesses more numerical isotropy and less dispersion error than that of the conventional explicit FD schemes.

REFERENCES

Bilbao, S., 2004, Wave and scattering methods for numerical simulation, Wiley.

Bilbao, S., Hamilton, B., 2013, Hexagonal vs. Rectilinear Grids for Explicit Finite Difference Schemes for the Two-dimensional Wave Equation, 21'st International Congress on Acoustics, Montréal, Canada.

Dudgeon, D.E., Mersereau, R.M., 1984, Multidimensional Digital Signal Processing, Prentice-Hall.

Fabero, J., Bautista, A., Casasús, L., 2001, An explicit finite differences scheme over hexagonal tessellation, Applied Mathematics Letters **14**, 593–598.

Smith, G.D., 1985, Numerical solution of partial differential equations: Finite difference methods, Clarendon Press.

Walstijn, M., Kowalczyk, K., 2008, On the numerical solution of the 2D wave equation with compact FDTD schemes, in Processing Digital Audio Effects, 205–212, Espoo, Finland.

Wu, W., Lines, L.R., Lu, H., 1996, Analysis of high-order, finite-difference schemes in 3-D reverse time migration, Geophysics, **61**, 845–856.