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Estimation of Stress-Strength Reliability for Stable Distributions

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Abstract

This paper deal with the estimation of Stress-Strength reliability parameter, $R = P(X < Y)$, when stress and strength are two independent stable distributions. The maximum likelihood estimator of stable distribution studied. Furthermore, we investigate the $R_{r,k} = P(X_{r:n_1} < Y_{k:n_2})$ for Lévy distribution as a member of stable family. Using a Monte Carlo simulation, the MSE and Bayes risk estimators are computed and compared.

Keywords: Stable distributions, Stress-Strength, Maximum likelihood estimator, Lindley approximation.

1 Introduction

Stable distributions are a class of probability distributions that specified by four parameters, an index of stability $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\gamma > 0$ and finally a location parameter $\delta \in \Re$. A stable distribution determined by its characteristic function, that is $X \sim S(\alpha, \beta, \gamma, \delta)$ if and only if $\varphi_X(t)$ as follows

$$\varphi_X(t) = \begin{cases} \exp\left\{-\gamma^\alpha |t|^\alpha \left[1 - i\beta \left(\tan\frac{\pi\alpha}{2}\right)(sign\, t)\right] + i\delta t\right\} & \alpha \neq 1, \\ \exp\left\{-\gamma\,|t| \left[1 + i\beta\frac{2}{\pi}(sign\, t)\log|t|\right] + i\delta t\right\} & \alpha = 1. \end{cases}$$

where $sign\, t$ is sign function, see Nolan [6].

Many authors discussed inference on R in reliability context. But there has not been much work on the estimation of R for Lévy distribution, the only paper, we are study is Ali and Woo [1]. In 2010 Eryilmaz [3] studied stress-strength reliability for a general coherent system. the probability $R_{r,k} = P(X_{r:n_1} < Y_{k:n_2})$ discussed by Pakdaman and Ahmadi [5].

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2 General case

To estimate Stress-Strength reliability for stable distributions we consider two cases:

I : X and Y have same and know skewness parameter, i.e. $X \sim S(\alpha_1, \beta)$ and $Y \sim S(\alpha_2, \beta)$ be independent random variables.

II : Skewness parameter be unknown, that is $X \sim S(\alpha_1, \beta_1)$ and $Y \sim S(\alpha_2, \beta_2)$ be independent random variables.

2.1 MLE

In case I we have

$$R = P(X < Y) = \int_{-\infty}^{+\infty} [1 - S_Y(z|\alpha_2, \beta)]\, s_X(z|\alpha_1, \beta)\, dz. \tag{1}$$

The S_Y and s_X are used to show the distribution function and density function of stable distributions, respectively. By computing the ML estimators of α_1 and α_2 we can calculate the (1) by a numerical method. Case II is same as I but we must obtain the ML estimators of α_1, β_1, α_2 and β_2. Simulation results are shown in Table 1.

3 Lévy distribution

Let X and Y be two independent random variables. In other word, $X \sim Lev(\alpha)$ and $Y \sim Lev(\beta)$ respectively. Ali and Woo [1] study $P(X < Y)$ for the Lévy distribution. In this paper, we investigate the case of $R_{r,k} = P(X_{r:n_1} < Y_{k:n_2})$.

Thus, the $R_{r,k}$ follows

$$R_{r,k} = P(X_{r:n_1} < Y_{k:n_2}) = \int_{-\infty}^{+\infty} F_{X_{r:n_1}}(z)\, f_{Y_{k:n_2}}(z)\, dz. \tag{2}$$

By formulas of pdf and cdf of the ith order statistic (See David and Nagaraja [2]) and writing the binomial expansion for $F_X^j(z)$ and $F_Y^{k-1}(z)$ we simplify $R_{r,k}$ as follows

$$\begin{aligned} R_{r,k} &= P(X_{r:n_1} < Y_{k:n_2}) \\ &= k \binom{k}{n_2} \sum_{j=r}^{n_1} \sum_{t=0}^{j} \sum_{l=0}^{k-1} \binom{n_1}{j} \binom{j}{t} \binom{k-1}{l} (-1)^{t+l} \\ &* \int_{-\infty}^{+\infty} [1 - F_X(z)]^{n_1 - t} [1 - F_Y(z)]^{n_2 - l - 1} f_Y(z)\, dz. \end{aligned} \tag{3}$$

3.1 MLE

Suppose $X_1, X_2, \cdots, X_{n_1}$ is a sample from $Lev(\alpha)$ and $Y_1, Y_2 \cdots, Y_{n_2}$ is a sample from $Lev(\beta)$, Calculate the likelihood function and by taking logarithm and derivative to α and β we obtain the maximum likelihood of parameters as follows

$$\hat{\alpha} = \frac{n_1}{\sum_{i=1}^{n_1} \frac{1}{x_i}}, \hat{\beta} = \frac{n_2}{\sum_{i=1}^{n_2} \frac{1}{y_i}}.$$

Since the maximum likelihood estimators have invariance property, we can calculate $R_{r,k}$ by numerical methods.

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3.2 Bayes estimator

Suppose the parameters, α and β, have the gamma priors, with following parameters

$$\alpha \sim GAM\,(k,\,\theta)\,, \quad and \ \ \beta \sim GAM\,(\mu,\sigma)\,.$$

Posterior pdfs of α and β are

$$\alpha|\underline{x} \sim GAM\left(\frac{n_1}{2}+k,\,\frac{1}{2}\sum_{i=1}^{n_1}\frac{1}{x_i}+\theta\right),\, and\ \beta|\underline{y} \sim GAM\left(\frac{n_2}{2}+\mu,\,\frac{1}{2}\sum_{i=1}^{n_1}\frac{1}{y_i}+\sigma\right).$$

3.3 Lindley's approximation

We consider Lindley's approximation (See Lindley [4]) from expanding about the posterior mode. Lindley's approximation leads to

$$\hat{U}_{Lindley} = \left(U(\theta)+\frac{1}{2}[B+Q_{30}B_{12}+Q_{21}C_{12}+Q_{12}C_{21}+Q_{03}B_{21}]\right)|_{(\theta_1,\theta_2)=(\tilde{\theta_1},\tilde{\theta_2})}, \qquad (4)$$

where $B = \sum_{i=1}^{2}\sum_{j=1}^{2}U_{ij}\tau_{ij}$ and $Q_{\eta\xi} = \frac{\partial^{\eta+\xi}}{\partial^{\eta}\theta_1\partial^{\xi}\theta_2}$ that $\eta,\xi = 0,1,2,3$. Furthermore, $i,j = 1,2$, $U_i = \frac{\partial U}{\partial\theta_i}$ and for $i \neq j$, $U_{ij} = \frac{\partial^2 U}{\partial\theta_i\partial\theta_j}$, $B_{ij} = (U_i\tau_{ii}+U_j\tau_{ij})\tau_{ii}$, $C_{ij} = 3U_i\tau_{ii}\tau_{ij} + U_j(\tau_{ii}\tau_{ij}+2\tau_{ij}^2)$.
Is the τ_{ij} (i,j)th element in the inverse of matrix $Q^* = (-Q^*_{ij}), i,j = 1,2$ so that $Q^*_{ij} = \frac{\partial^2 Q}{\partial\theta_i\partial\theta_j}$.

It is not difficult to obtain the above terms.

4 Simulation study

Table 1 shows the bias and MSE of R in general case of a stable distribution. Furthermore,

Table 1: Bias and MSE of R for stable law with unknown β

	Symmetric				Positive				Asymmetric			
$n_1 = n_2$	α	β	Bias	MSE	α	β	Bias	MSE	α	β	Bias	MSE
5	0.3	0	-0.0396	0.0973	0.2	1	0.0738	0.0288	1.2	1	0.0793	0.1223
	0.8	0	-0.0094	0.1427	0.4	1	-0.0120	0.0472	1.4	1	0.0028	0.0496
	1.2	0	-0.0801	0.0529	0.7	1	-0.0226	0.0060	1.7	1	-0.0119	0.0695
	2	0	-0.0025	0.0151	0.9	1	-0.0041	0.0025	2	1	0.0174	0.0171
10	0.3	0	-0.0335	0.827	0.2	1	-0.0153	0.0203	1.2	1	0.0019	0.1091
	0.8	0	0.0999	0.0951	0.4	1	0.0451	0.0020	1.4	1	0.0013	0.0765
	1.2	0	0.1120	0.0970	0.7	1	0.0117	0.0142	1.7	1	-0.0036	0.0370
	2	0	-0.0108	0.0101	0.9	1	0.1056	0.0401	2	1	0.0391	0.0167

we simulate the Estimated Risks (ER) of Bayes and approximation Bayes estimators with respect to prior parameters.

Conclusion

We have used the symmetric, positive asymmetric and asymmetric stable laws for simulating bias and MSE. We have observed that the Bayes estimator has the smallest estimated risk. The estimated risk decreases as the priors become more "informative".

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Table 2: Bias and ER for Bayes Estimators of $R_{n_1,1}$

Informative	k	θ	μ	σ	$n_1 = n_2$	ER(R^B_{Lin})	Bias(R^B_{Lin})	ER(R^B)	Bias(R^B)
Least informative	1	2	1	2	5	0.320	0.562	0.358	0.583
	1	2	1	2	10	0.264	-0.513	0.273	0.508
Informative	5	10	5	10	5	0.273	0.503	0.289	0.519
	5	10	5	10	10	0.203	-0.467	0.229	-0.471
Most informative	10	20	10	20	5	0.241	0.488	0.251	0.497
	10	20	10	20	10	0.191	-0.438	0.197	-0.439

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