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Baysian Estimation of Lifetime Performance Index Based on RSS Sample

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Abstract

Lifetime performance index (C_L) is a flexible and effective tool for evaluating product quality and conforming rate. Ranked set sampling (RSS) scheme is applied for Baysian estimator of C_L based on square error loss. We assume that lifetimes of products follow a one-parameter exponential distribution. The simulation result for this scheme is compared with simple random sample (SRS) scheme based on bias, risk, pitman nearness, relative efficiency.

Keywords: Ranked set sampling, Lifetime performance index, Baysian estimation

1 Introduction

In lifetime testing experiments, the experimenter because of time limitation or other restrictions such as lack of funds, lack of material resources, mechanical or experimental difficulties, etc on data collection, may not always be in a position to observe the lifetimes of all the products on test. In this paper, we propose sampling scheme known as ranked-set sampling (RSS), introduced by McIntyre [3], instead of simple random sample (SRS) for estimating and testing a lifetime performance index C_L, since this method requires fewer observations to provide the same information[1] . C_L index, proposed by Montgomery [4], has many applications in health care and public health monitoring and surveillance and used to measure the larger-the-better type quality characteristics. This index defined as: $C_L = \frac{\mu - L}{\sigma}$, where μ is the process mean, σ is the process standard deviation and L is the lower specification limit. The ratio of conforming products is known as the conforming rate and can be defined as $p = P(X \geq L) = e^{C_L - 1}, -\infty < C_L < 1$. Obviously, a strictly

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increasing relationship exists between the conforming rate p and the lifetime performance index C_L. C_L and θ can be derived as $C_L = 1 + \ln p$ and $\theta = -\frac{L}{\ln p}$ respectively. The RSS scheme can be used for hospital monitoring with respect to patient infection rates, patient falls or accidents, emergency waiting room times, and so on. The data from patients can be obtained via RSS schemes using expert's knowledge or using auxiliary variables [2]. In this paper we assume that the lifetime data follow a one-parameter exponential distribution, $\varepsilon(\theta)$, with pdf $f(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$. In this case, the capability index C_L reduces to $C_L = 1 - \frac{L}{\theta}$. To obtain a ranked set sampling, suppose $X_1, X_2, ..., X_n$ be a random sample of size n with pdf $f(x)$ and we have n set of such sample. Ranked set sampling is sourced by the sample selection which is based on two stages, involves an initial ranking samples of size n as follows:

Table 1: Ranking the samples

1	$X_{1(1)}$	$X_{1(2)}$	$\dots$	$X_{1(n-1)}$	$X_{1(n)}$
2	$X_{2(1)}$	$X_{2(2)}$	$\dots$	$X_{2(n-1)}$	$X_{2(n)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$X_{n(1)}$	$X_{n(2)}$	$\dots$	$X_{n(n-1)}$	$X_{n(n)}$

Here, $X_{i(j)}$,$(i, j = 1 : n)$ denotes the jth order statistic of the ith random sample. RSS sample is formed by selecting the diagonal elements in Table 1. Element of new sample RSS are independent but not identically distributed. In certain situations, the whole procedure to generate an RSS of size n can be repeated m times. Success of RSS depends very much on our ability to rank the units without any error.

2 Baysian estimatin for C_L based on RSS

In this section, based on two different prior, $IG(a, b)$ with known parameters a, b and Jeffrey's prior, we obtain Bayes estimators for C_L based on RSS samples and study performance of these estimators.

2.1 Inverse Gamma prior

Suppose $\theta \sim IG(a, b)$, which probability density function is defined as $\pi(\theta) = \frac{b^a}{\Gamma(a)}(\frac{1}{\theta})^{a+1}e^{-\frac{b}{\theta}}$. By Bayes' theorem, the posterior distribution of θ, $\pi(\theta|X_{srs})$, is $IG(n+a, b+\sum x_i)$. Therefore the Bayes estimators for θ under square error loss is equal to $E(\theta|X_{srs}) = \frac{b+n\bar{X}}{n+a-1}$, and Bayes estimator for C_L will be obtained $\hat{C}^{srs}_{Lbayes-IG} = 1 - LE(\frac{1}{\theta}|X_{srs}) = 1 - L(\frac{n+a}{b+n\bar{X}})$. Let $X_{rss} = \{X_{(1,1)}, X_{(2,2)}, \dots, X_{(m,m)}\}$ is one-cycle RSS sample from $\varepsilon(\theta)$. The joint probablity density function of the RSS, due to the independence of element X_{rss}, is given by Sadek et al. [5] such as

$$g_\theta(X_{rss}) = \sum_{j_1=0}^{0}\sum_{j_2=0}^{1}\dots\sum_{j_n=0}^{m-1}\prod_{i=1}^{m}(\frac{1}{\theta})^m e^{-\frac{1}{\theta}\sum_{i=1}^{m}(n+k-i+1)}x(i, i). \quad (1)$$

So the posterior density can be written as

$$\pi(\theta|X_{rss})=\frac{\sum_{j_1=0}^{0}\sum_{j_2=0}^{1}\cdots\sum_{j_n=0}^{m-1}\prod_{i=1}^{m}C_{j_i}(i)(\frac{1}{\theta})^{m+a+1}e^{-\frac{1}{\theta}(b+\sum_{i=1}^{m}(n+j_i-i+1)x_{(i,i)})}}{\sum_{j_1=0}^{0}\sum_{j_2=0}^{1}\cdots\sum_{j_m=0}^{m-1}\prod_{i=1}^{m}C_{j_k}(i)(b+\sum_{i=1}^{m}(m+j_i-i+1)(x_{(i,i)})^{-(m+a)}\Gamma(m+a)}$$

Then the Bayes estimator of C_L is $\hat{C}^{rss}_{Lbayes_{IG}}=1-LE(\frac{1}{\theta}|X_{rss})=1-L\int_0^\infty\frac{1}{\theta}\pi(\theta|X_{rss})d\theta$

2.2 Jeffry's prior

If θ has the Jeffrey prior, $\pi(\theta)\propto\frac{1}{\theta}$, then $\frac{1}{\theta}|X_{srs}\sim G(n,\frac{1}{\sum X_i})$. Therefore, the Bayes estimator of C_L is $E(C_L|X_{srs})=1-L\frac{n}{\sum X_i}=1-\frac{L}{\bar{X}_{srs}}$.
In the case of RSS scheme

$$\pi(\theta|X_{rss})=\frac{\sum_{j_1=0}^{0}\sum_{j_2=0}^{1}\cdots\sum_{j_n=0}^{m-1}\prod_{i=1}^{m}C_{j_i}(i)(\frac{1}{\theta})^{m+1}e^{-\frac{1}{\theta}(\sum_{i=1}^{m}(m+j_i-i+1)x_{(i,i)})}}{\sum_{j_1=0}^{0}\sum_{j_2=0}^{1}\cdots\sum_{j_n=0}^{m-1}\prod_{i=1}^{m}C_{j_k}(i)(\sum_{i=1}^{m}(m+j_i-i+1)(x_{(i,i)})^{-(m)}\Gamma(m)},$$

therefore $\hat{C}^{rss}_{Lbayes_J}=1-LE(\frac{1}{\theta}|X_{rss})=1-\int_0^\infty\frac{1}{\theta}\pi(\theta|X_{rss})d\theta$

2.3 Simulation study

For studying performance of discussed estimators, we carry out Monte-Carlo simulations as follows:

1. Determine lower specification limit L, hyper parameter (a,b), sample size n. Generate θ_0 from distribution $IG(a,b)$ and calculate C_L. 10000 times repeat steps 2, 3.

2. Generate SRS and RSS samples of size n from $\varepsilon(\theta_0)$ and derived $\hat{C}^{srs}_{Lbayes_{IG}}$, $\hat{C}^{srs}_{Lbayes_J}$, $\hat{C}^{rss}_{Lbayes_{IG}}$, $\hat{C}^{rss}_{Lbayes_J}$.

3. For each samples and estimators in the step (2), calculate, $di=(\hat{C}_{Li}-C_L), i=1,...,10000$. In each times, for calculating the Pitman nearness criteria between two estimators, we investigate if $|\hat{C}_{L1}-C_L|<|\hat{C}_{L2}-C_L|$.

4. The risk values of $\hat{C}_{Li}$ is the mean of di^2. Relative Efficiency between $\hat{C}_{L1}$ and $\hat{C}_{L2}$ is $RE(\hat{C}_{L1},\hat{C}_{L2})=\frac{MSE(\hat{C}_{L1})}{MSE(\hat{C}_{L2})}$. The bias of $\hat{C}_L$ is $\frac{1}{10000}\left(\sum_{i=1}^{10000}\hat{C}_{Li}\right)-C_L$. The pitman nearness between $\hat{C}_{L1}$ and $\hat{C}_{L2}$ is $\frac{1}{10000}\#|\hat{C}_{L1}-C_L|<|\hat{C}_{L2}-C_L|$.

Example 1. *We select values of hyper parameters, (a,b), in prior distribution, such that the mean of prior distribution, $IG(a,b)$, is fixed at 0.5 and for it's variance we consider three state: small (0.0357), moderate (0.0833) and large (0.25). With this strategy select $(a,b)=(9,4),(5,2),(3,1)$. Let $L=1.04$ and $n=4,5,6$. Table 2 shows the values of Bias, Risk, Relative Efficiency (RE) and Pitman Nearness criterion (PN).*

Table 2: Observed values of Bias, Risk, RE, PN

				IGestimator		jeffrys estimator			pitman-nearness		Relative efficiency	
4	3	1	Bias	0.0347	0.0187	-1.0689	-0.3888	PN	0.4103	0.3779	0.6346	0.5873
			Risk	1.6624	1.0309	15.718	2.429	RE	1.6126	6.3284	0.1081	0.4244
	5	2	Bias	-0.0145	-0.0099	-0.8825	-0.3292	PN	0.4115	0.3727	0.6676	0.6083
			Risk	0.8147	0.5279	8.512	1.4876	RE	1.5434	5.7221	0.0957	0.3549
	9	4	Bias	0.0085	0.0012	-0.7505	-0.2963	PN	0.4306	0.3876	0.7057	0.6446
			Risk	0.437	0.3231	5.525	1.1639	RE	1.3525	4.747	0.0791	0.2776
5	3	1	Bias	0.0157	0.0013	-0.7643	-0.2641	PN	0.3906	0.359	0.6144	0.5795
			Risk	1.4285	0.7502	7.3862	1.4537	RE	1.9042	0.1016	5.0810	0.7634
	5	2	Bias	-0.0088	-0.0001	-0.6944	-0.2202	PN	0.3998	0.3517	0.6564	0.602
			Risk	0.7473	0.411	4.8278	0.9213	RE	1.8182	5.2402	0.1548	0.4461
	9	4	Bias	0.0055	0.0019	-0.5689	-0.1889	PN	0.4113	0.3575	0.691	0.6207
			Risk	0.4117	0.2602	3.3125	0.6376	RE	1.5822	5.1953	0.1243	0.4081
6	3	1	Bias	-0.0115	0.0046	-0.69	-0.1802	PN	0.3624	0.3349	0.6117	0.5558
			Risk	1.3246	0.5905	6.2285	0.93	RE	2.2431	6.6974	0.2127	0.6350
	5	2	Bias	0.0047	0.0041	-0.5264	-0.1544	PN	0.3823	0.3356	0.6404	0.5796
			Risk	0.6953	0.3356	3.1049	0.5839	RE	2.0718	5.3175	0.2239	0.5748
	9	4	Bias	0.0007	0	-0.4829	-0.1407	PN	0.3898	0.331	0.6772	0.6049
			Risk	0.3849	0.2114	2.489	0.4472	RE	1.8211	5.5656	0.1547	0.4727

Table 2 shows that absolute values of bias and also risk for $\hat{C}^{rss}_{Lbayes-IG}$ and $\hat{C}^{rss}_{Lbayes-J}$ are smaller than similar estimators in SRS scheme.

Moreover the RE and PN probability criteria indicates the efficiency of RSS estimators with respect to SRS estimators. Because of reducing the cost of data collection and better performance estimators in simulation for RSS scheme, we suggest that $\hat{C}^{rss}_{Lbayes-IG}$ and $\hat{C}^{rss}_{Lbayes-J}$ estimators as long as there are no ranking errors caused by a large set of size m.

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