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On the Number of Failed Links in a Three-State Network

Ashrafi, S.¹

Department of Statistics, University of Isfahan

Abstract

In this paper, we consider a single-step network consists of n links and assume that the links are subject to failure. It is assumed that the network can be in three states, up (K = 2), partial performance (K = 1) and down (K = 0). Under different scenarios on the states of the network and using the concept of two-dimensional signature, we obtain the probabilities that i links fail at time t_1 and j links fail at time t. Several stochastic and aging properties of the proposed probabilities are studied.

Keywords: Signature matrix, Bivariate increasing failure rate, Total positive of order 2, Stochastic order.

1 Introduction

In this paper, we consider a three-state network consisting of n i.i.d. binary links. We assume that the network can be in three states, up (K = 2), partial performance (K = 1) and down (K = 0). Let the network start to function at time t = 0 in state K = 2. Denote by T_1 the lifetime of the network which remains in state K = 2. Also, denote by T the network lifetime i.e. the entrance time into state K = 0. Using these notations, the two-dimensional signature of the network is defined to be a probability matrix S with elements defined by

$$s_{i,j} = \frac{n_{i,j}}{n!}, \ 1 \le i < j \le n,$$

where $n_{i,j}$ is the number of ways that the *i*th and the *j*th links failure cause the state of the network changes from K = 2 to K = 1 and from K = 1 to K = 0, respectively.

Recently Erylmaz (2010) studied the distribution and expected value of the number of working components at time t in a consecutive k-out-of-n system under the condition that it is working at time t. Asadi and Berred (2012) studied the number of failed components in a binary coherent system. In this paper, we assume that at time t_1 the network is

¹s.ashrafi@ui.ac.ir

in state K = 2 and at time t, it is in state K = 1 or it is functioning. Then, we present a model for the probabilities that k and l, $0 \le k < l \le n-1$ links have failed at times t_1 and t, respectively. Based on the notion two-dimensional signature, we obtain some stochastic and aging properties of the proposed probabilities.

2 Main results

Consider a network consisting of n links. Suppose that $X_1, ..., X_n$ denote the links lifetimes, where we assume that X_i 's are i.i.d with a common continuous distribution function F(x). Suppose that, we have some information about the states of the network at times t_1 and $t, t_1 < t$, for example, we know $T_1 \in A_1$ and $T \in A$ where $A_1, A \subseteq [0, \infty)$. Denote by N(t)the number of failed links in [0, t]. In such a situation, we are interested in the conditional probability

$$p_{A_1,A}(k,l) = P(N(t_1) = k, N(t) = l | T_1 \in A_1, T \in A), \quad 0 \le k \le l \le n.$$

In this paper, we consider two following cases:

(I) Suppose that at time t_1 the network is in state K = 2 and at time $t, t > t_1$, it is in state K = 1. In such a situation $A_1 = (t_1, t)$ and $A = (t, \infty)$. In this case, $p_{A_1,A}(k, l)$, which we denote it by $p_{t_1,t}(k, l)$, is

$$p_{t_1,t}(k,l) = P(N(t_1) = k, N(t) = l | t_1 < T_1 < t, T > t), \quad 0 \le k < l \le n - 1.$$

(II) Suppose that at time t_1 network is in state k = 2, and at time t, it is functioning. In such a situation, $A_1 = (t_1, \infty)$ and $A = (t, \infty)$. In this case, $p_{A_1,A}(k, l)$, which we denote it by $q_{t_1,t}(k, l)$, is

$$q_{t_1,t}(k,l) = P(N(t_1) = k, N(t) = l | T_1 > t_1, T > t), \quad 0 \le k \le l \le n - 1.$$

In the following theorem, $p_{t_1,t}(k,l)$ and $q_{t_1,t}(k,l)$ are computed.

Theorem 1. Consider a network consists of n links with i.i.d. lifetimes. Suppose that F(x) denotes the common distribution of the links lifetimes and T_1 and T are the lifetime in state K = 2 and the lifetime of the network, respectively. Assume that S is the signature matrix of the network.

(a) If $\beta_{k,l} = \sum_{i=k+1}^{l} \sum_{j=l+1}^{n} s_{i,j}$ then

$$p_{t_1,t}(k,l) = \frac{\beta_{k,l}c_{k,l,n}F^k(t_1)(F(t) - F(t_1))^{l-k}\bar{F}^{n-l}(t)}{\sum_{k=0}^{n-2}\sum_{l=k+1}^{n-1}\beta_{k,l}c_{k,l,n}F^k(t_1)(F(t) - F(t_1))^{l-k}\bar{F}^{n-l}(t)}, \ 0 \le k < l \le n-1$$

where $c_{k,l} = \frac{n!}{k!(l-k)!(n-l)!}$.

(b) If $\bar{S}_{k,l} = \sum_{i=k+1}^{l} \sum_{j=\max\{i,l\}+1}^{n} s_{i,j}$ then

$$q_{t_{1,t}}(k,l) = \frac{c_{k,l,n}S_{k,l}F^{k}(t_{1})(F(t) - F(t_{1}))^{l-k}F^{n-l}(t)}{\sum_{i=1}^{n-1}\sum_{j=i}^{n}c_{i,j,n}\bar{S}_{i,j}F^{i}(t_{1})(F(t) - F(t_{1}))^{j-i}\bar{F}^{n-j}(t)}, \ 0 \le k \le l \le n-1.$$

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In the following, we present results that compare the probabilities of the number of failed links of two networks. Before it, we need the following definition.

Definition 1. Let $f_1(x, y)$ and $f_2(x, y)$ be two nonnegative functions. $f_1(x, y)$ is said to be smaller than $f_2(x, y)$ in the total positive order (denoted by $f_1 \leq_{TP_2} f_2$) if $f_1(\mathbf{x})f_2(\mathbf{y}) \leq$ $f_1(\mathbf{x} \wedge \mathbf{y})f_2(\mathbf{x} \vee \mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in R^2$, where $\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \min\{x_2, y_2\})$ and $\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \max\{x_2, y_2\}).$

Theorem 2. Consider two networks each consists of n i.i.d. links. Suppose that the links lifetimes of two networks have the same distribution. Let S_1 and S_2 be the corresponding signature matrices and $\beta_{k,l}^{(r)} = \sum_{i=k+1}^{l} \sum_{j=l+1}^{n} s_{r,i,j}$ and $\bar{S}_{k,l}^{(r)} = \sum_{i=k+1}^{l} \sum_{j=\max\{i,l\}+1}^{n} s_{r,i,j}, r = 1, 2$. Suppose that $p_{t_1,t}^{(r)}(k,l)$ and $q_{t_1,t}^{(r)}(k,l)$ are the probability functions corresponding to $\beta_{k,l}^{(r)}$ and $\bar{S}_{k,l}^{(r)}$, r = 1, 2, respectively.

(a) If $\beta_{k,l}^{(1)} \leq_{TP_2} \beta_{k,l}^{(2)}$ then $p_{t_1,t}^{(1)}(k,l) \leq_{TP_2} p_{t_1,t}^{(2)}(k,l)$. (b) If $\bar{S}_{k,l}^{(1)} \leq_{TP_2} \bar{S}_{k,l}^{(2)}$ then $q_{t_1,t}^{(1)}(k,l) \leq_{TP_2} q_{t_1,t}^{(2)}(k,l)$.

Recall that if in Definition 1, f_1 and f_2 are probability mass functions of (X_1, X_2) and (Y_1, Y_2) , respectively, then TP_2 order is called likelihood ratio order and denoted by $(X_1, X_2) \leq_{lr} (Y_1, Y_2)$.

In the following theorem, under some stochastic comparisons between links lifetimes of two networks, we compare the probabilities of the number of failed links of two networks.

Theorem 3. Consider two networks each consists of n i.i.d. links. Assume that two networks have the same structure and F_1 and F_2 are the corresponding distributions of the link lifetimes. Suppose that $p_{t_1,t}^{(i)}(k,l)$ and $q_{t_1,t}^{(i)}(k,l)$ are the probability functions corresponding to F_i , i = 1, 2. Let $(I_1^{(i)}, I_2^{(i)})$ and $(J_1^{(i)}, J_2^{(i)})$ have joint probability mass functions $p_{t_1,t}^{(i)}(k,l)$ and $q_{t_1,t}^{(i)}(k,l)$, i = 1, 2, respectively. If $F_1 \leq_{rh} F_2$, $F_1 \leq_{hr} F_2$ and

- (a) $\beta_{k,l}$ is TP_2 in k and l then $(I_1^{(1)}, I_2^{(1)}) \ge_{lr} (I_1^{(2)}, I_2^{(2)}).$
- (b) $\bar{S}_{k,l}$ is TP_2 in k and l then $(J_1^{(1)}, J_2^{(1)}) \ge_{lr} (J_1^{(2)}, J_2^{(2)})$.

The following definition is an analogue to that of Harris (1970) in the continuous set up.

Definition 2. The bivariate mass function $p_{i,j}$ with survival function $\bar{P}_{i,j}$ is said to be BIFR if $\bar{P}_{i,j}$ is TP_2 and $\frac{\bar{P}_{i+1,j+1}}{\bar{P}_{i,j}}$ is decreasing in i, j.

Theorem 4. Let $\bar{P}_{t_1,t}(k,l)$ and $\bar{Q}_{t_1,t}(k,l)$ be the survival functions corresponding to probability mass functions $p_{t_1,t}(k,l)$ and $q_{t_1,t}(k,l)$, respectively.

- (a) If $\beta_{k,l}$ is TP_2 in k and l and $\frac{\beta_{k+1,l+1}}{\beta_{k,l}}$ is decreasing in k and l then $\bar{P}_{t_1,t}(k,l)$ is BIFR.
- (b) If $\bar{S}_{k,l}$ is TP_2 in k and l and $\frac{\bar{S}_{k+1,l+1}}{\bar{S}_{k,l}}$ is decreasing in k and l then $\bar{Q}_{t_1,t}(k,l)$ is BIFR.

The following example present an application of Theorem 4.

Example 1. Figure ?? presents a network consists of 5 nodes and 10 links. Assume that links are subject to failures. The states of the network are defined as K = 2 if all nodes are connected, K = 1 if nodes are divided into two disconnected sets, and K = 0 if nodes are divided into at least three disconnected sets.



Figure 1: Network with 5 nodes and 10 links

The signature matrix (S) of this network is given in Gertsbakh and Shpungin (2012). It can be seen that $\beta_{k,4} = 0.0241$, $\beta_{k,5} = 0.1183$, $\beta_{k,6} = 0.4049$, $\beta_{k,7} = 0.9166$, k = 0, ..., 3 and $\beta_{4,5} = 0.0942$, $\beta_{4,6} = 0.3808$, $\beta_{4,7} = 0.8972$, $\beta_{5,6} = 0.2866$, $\beta_{5,7} = 0.8221$, $\beta_{6,7} = 0.5951$. It can be shown that $\beta_{k,l}$ is TP_2 in k and l and $\frac{\beta_{k+1,l+1}}{\beta_{k,l}}$ is decreasing in k and l.

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