First Seminar on
Reliability Theory and its Applications
27-28, May 2015

On the Number of Failed Links in a Three-State Network

Ashrafi, S. [1]

Department of Statistics, University of Isfahan

Abstract

In this paper, we consider a single-step network consists of n links and assume that the links are subject to failure. It is assumed that the network can be in three states, *up* $(K = 2)$, *partial performance* $(K = 1)$ and *down* $(K = 0)$. Under different scenarios on the states of the network and using the concept of two-dimensional signature, we obtain the probabilities that i links fail at time t_1 and j links fail at time t . Several stochastic and aging properties of the proposed probabilities are studied.

Keywords: Signature matrix, Bivariate increasing failure rate, Total positive of order 2, Stochastic order.

1 Introduction

In this paper, we consider a three-state network consisting of n i.i.d. binary links. We assume that the network can be in three states, *up* $(K = 2)$, *partial performance* $(K = 1)$ and *down* $(K = 0)$. Let the network start to function at time $t = 0$ in state $K = 2$. Denote by T_1 the lifetime of the network which remains in state $K = 2$. Also, denote by T the network lifetime i.e. the entrance time into state $K = 0$. Using these notations, the two-dimensional signature of the network is defined to be a probability matrix S with elements defined by

$$s_{i,j} = \frac{n_{i,j}}{n!}, \quad 1 \leq i < j \leq n,$$

where $n_{i,j}$ is the number of ways that the ith and the jth links failure cause the state of the network changes from $K = 2$ to $K = 1$ and from $K = 1$ to $K = 0$, respectively.

Recently Erylmaz (2010) studied the distribution and expected value of the number of working components at time t in a consecutive k-out-of-n system under the condition that it is working at time t. Asadi and Berred (2012) studied the number of failed components in a binary coherent system. In this paper, we assume that at time t_1 the network is

[1] s.ashrafi@ui.ac.ir

in state $K = 2$ and at time t, it is in state $K = 1$ or it is functioning. Then, we present a model for the probabilities that k and l, $0 \leq k < l \leq n-1$ links have failed at times t_1 and t, respectively. Based on the notion two-dimensional signature, we obtain some stochastic and aging properties of the proposed probabilities.

2 Main results

Consider a network consisting of n links. Suppose that $X_1, ..., X_n$ denote the links lifetimes, where we assume that X_i's are i.i.d with a common continuous distribution function $F(x)$. Suppose that, we have some information about the states of the network at times t_1 and t, $t_1 < t$, for example, we know $T_1 \in A_1$ and $T \in A$ where $A_1, A \subseteq [0, \infty)$. Denote by $N(t)$ the number of failed links in $[0, t]$. In such a situation, we are interested in the conditional probability

$$p_{A_1,A}(k,l) = P(N(t_1) = k, N(t) = l | T_1 \in A_1, T \in A), \quad 0 \leq k \leq l \leq n.$$

In this paper, we consider two following cases:

(I) Suppose that at time t_1 the network is in state $K = 2$ and at time t, $t > t_1$, it is in state $K = 1$. In such a situation $A_1 = (t_1, t)$ and $A = (t, \infty)$. In this case, $p_{A_1,A}(k,l)$, which we denote it by $p_{t_1,t}(k,l)$, is

$$p_{t_1,t}(k,l) = P(N(t_1) = k, N(t) = l | t_1 < T_1 < t, T > t), \quad 0 \leq k < l \leq n-1.$$

(II) Suppose that at time t_1 network is in state $k = 2$, and at time t, it is functioning. In such a situation, $A_1 = (t_1, \infty)$ and $A = (t, \infty)$. In this case, $p_{A_1,A}(k,l)$, which we denote it by $q_{t_1,t}(k,l)$, is

$$q_{t_1,t}(k,l) = P(N(t_1) = k, N(t) = l | T_1 > t_1, T > t), \quad 0 \leq k \leq l \leq n-1.$$

In the following theorem, $p_{t_1,t}(k,l)$ and $q_{t_1,t}(k,l)$ are computed.

Theorem 1. *Consider a network consists of n links with i.i.d. lifetimes. Suppose that $F(x)$ denotes the common distribution of the links lifetimes and T_1 and T are the lifetime in state $K = 2$ and the lifetime of the network, respectively. Assume that S is the signature matrix of the network.*

(a) If $\beta_{k,l} = \sum_{i=k+1}^{l} \sum_{j=l+1}^{n} s_{i,j}$ then

$$p_{t_1,t}(k,l) = \frac{\beta_{k,l} c_{k,l,n} F^k(t_1)(F(t) - F(t_1))^{l-k} \bar{F}^{n-l}(t)}{\sum_{k=0}^{n-2} \sum_{l=k+1}^{n-1} \beta_{k,l} c_{k,l,n} F^k(t_1)(F(t) - F(t_1))^{l-k} \bar{F}^{n-l}(t)}, \quad 0 \leq k < l \leq n-1$$

where $c_{k,l} = \frac{n!}{k!(l-k)!(n-l)!}$.

(b) If $\bar{S}_{k,l} = \sum_{i=k+1}^{l} \sum_{j=\max\{i,l\}+1}^{n} s_{i,j}$ then

$$q_{t_1,t}(k,l) = \frac{c_{k,l,n} \bar{S}_{k,l} F^k(t_1)(F(t) - F(t_1))^{l-k} \bar{F}^{n-l}(t)}{\sum_{i=1}^{n-1} \sum_{j=i}^{n} c_{i,j,n} \bar{S}_{i,j} F^i(t_1)(F(t) - F(t_1))^{j-i} \bar{F}^{n-j}(t)}, \quad 0 \leq k \leq l \leq n-1.$$

In the following, we present results that compare the probabilities of the number of failed links of two networks. Before it, we need the following definition.

Definition 1. Let $f_1(x,y)$ and $f_2(x,y)$ be two nonnegative functions. $f_1(x,y)$ is said to be smaller than $f_2(x,y)$ in the total positive order (denoted by $f_1 \leq_{TP_2} f_2$) if $f_1(\mathbf{x})f_2(\mathbf{y}) \leq f_1(\mathbf{x} \wedge \mathbf{y})f_2(\mathbf{x} \vee \mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in R^2$, where $\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \min\{x_2, y_2\})$ and $\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \max\{x_2, y_2\})$.

Theorem 2. *Consider two networks each consists of n i.i.d. links. Suppose that the links lifetimes of two networks have the same distribution. Let S_1 and S_2 be the corresponding signature matrices and $\beta^{(r)}_{k,l} = \sum_{i=k+1}^{l}\sum_{j=l+1}^{n} s_{r,i,j}$ and $\bar{S}^{(r)}_{k,l} = \sum_{i=k+1}^{l}\sum_{j=\max\{i,l\}+1}^{n} s_{r,i,j}, r = 1,2$. Suppose that $p^{(r)}_{t_1,t}(k,l)$ and $q^{(r)}_{t_1,t}(k,l)$ are the probability functions corresponding to $\beta^{(r)}_{k,l}$ and $\bar{S}^{(r)}_{k,l}$, $r = 1,2$, respectively.*

(a) If $\beta^{(1)}_{k,l} \leq_{TP_2} \beta^{(2)}_{k,l}$ then $p^{(1)}_{t_1,t}(k,l) \leq_{TP_2} p^{(2)}_{t_1,t}(k,l)$.

(b) If $\bar{S}^{(1)}_{k,l} \leq_{TP_2} \bar{S}^{(2)}_{k,l}$ then $q^{(1)}_{t_1,t}(k,l) \leq_{TP_2} q^{(2)}_{t_1,t}(k,l)$.

Recall that if in Definition 1, f_1 and f_2 are probability mass functions of (X_1, X_2) and (Y_1, Y_2), respectively, then TP_2 order is called likelihood ratio order and denoted by $(X_1, X_2) \leq_{lr} (Y_1, Y_2)$.

In the following theorem, under some stochastic comparisons between links lifetimes of two networks, we compare the probabilities of the number of failed links of two networks.

Theorem 3. *Consider two networks each consists of n i.i.d. links. Assume that two networks have the same structure and F_1 and F_2 are the corresponding distributions of the link lifetimes. Suppose that $p^{(i)}_{t_1,t}(k,l)$ and $q^{(i)}_{t_1,t}(k,l)$ are the probability functions corresponding to F_i, $i = 1,2$. Let $(I^{(i)}_1, I^{(i)}_2)$ and $(J^{(i)}_1, J^{(i)}_2)$ have joint probability mass functions $p^{(i)}_{t_1,t}(k,l)$ and $q^{(i)}_{t_1,t}(k,l)$, $i = 1,2$, respectively. If $F_1 \leq_{rh} F_2$, $F_1 \leq_{hr} F_2$ and*

(a) $\beta_{k,l}$ is TP_2 in k and l then $(I^{(1)}_1, I^{(1)}_2) \geq_{lr} (I^{(2)}_1, I^{(2)}_2)$.

(b) $\bar{S}_{k,l}$ is TP_2 in k and l then $(J^{(1)}_1, J^{(1)}_2) \geq_{lr} (J^{(2)}_1, J^{(2)}_2)$.

The following definition is an analogue to that of Harris (1970) in the continuous set up.

Definition 2. The bivariate mass function $p_{i,j}$ with survival function $\bar{P}_{i,j}$ is said to be $BIFR$ if $\bar{P}_{i,j}$ is TP_2 and $\frac{\bar{P}_{i+1,j+1}}{\bar{P}_{i,j}}$ is decreasing in i,j.

Theorem 4. *Let $\bar{P}_{t_1,t}(k,l)$ and $\bar{Q}_{t_1,t}(k,l)$ be the survival functions corresponding to probability mass functions $p_{t_1,t}(k,l)$ and $q_{t_1,t}(k,l)$, respectively.*

(a) If $\beta_{k,l}$ is TP_2 in k and l and $\frac{\beta_{k+1,l+1}}{\beta_{k,l}}$ is decreasing in k and l then $\bar{P}_{t_1,t}(k,l)$ is $BIFR$.

(b) If $\bar{S}_{k,l}$ is TP_2 in k and l and $\frac{\bar{S}_{k+1,l+1}}{\bar{S}_{k,l}}$ is decreasing in k and l then $\bar{Q}_{t_1,t}(k,l)$ is $BIFR$.

The following example present an application of Theorem 4.

Example 1. Figure ?? presents a network consists of 5 nodes and 10 links. Assume that links are subject to failures. The states of the network are defined as $K = 2$ if all nodes are connected, $K = 1$ if nodes are divided into two disconnected sets, and $K = 0$ if nodes are divided into at least three disconnected sets.

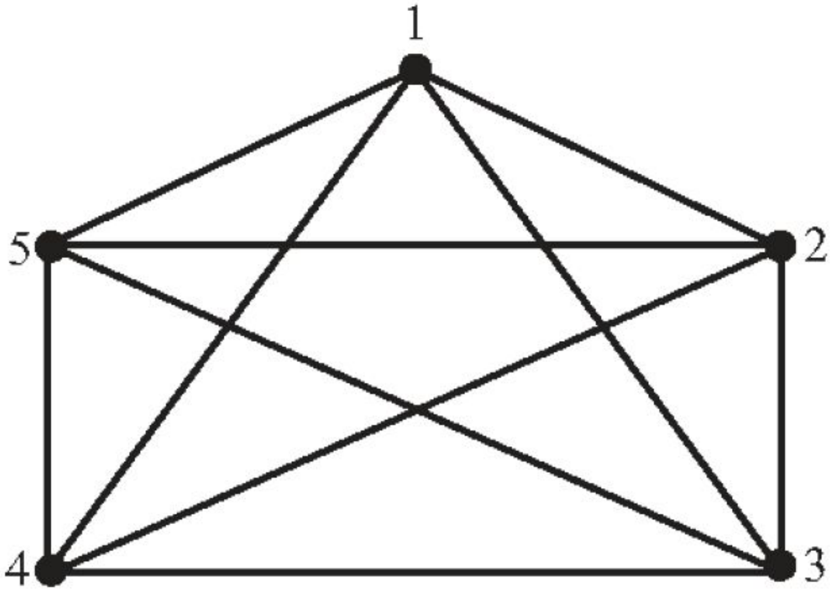

Figure 1: Network with 5 nodes and 10 links

The signature matrix (S) of this network is given in Gertsbakh and Shpungin (2012). It can be seen that $\beta_{k,4} = 0.0241,\ \beta_{k,5} = 0.1183,\ \beta_{k,6} = 0.4049,\ \beta_{k,7} = 0.9166,\ k = 0, ..., 3$ and $\beta_{4,5} = 0.0942,\ \beta_{4,6} = 0.3808,\ \beta_{4,7} = 0.8972,\ \beta_{5,6} = 0.2866,\ \beta_{5,7} = 0.8221,\ \beta_{6,7} = 0.5951$. It can be shown that $\beta_{k,l}$ is TP_2 in k and l and $\frac{\beta_{k+1,l+1}}{\beta_{k,l}}$ is decresing in k and l.

References

[1] Asadi, M. and Berred, A. (2012), On the number of failed components in a coherent operating system. *Statistics & Probability Letters*, **82**(12), 2156-2163.

[2] Eryilmaz, S. (2010), Number of working components in consecutive k-out-of-n system while it is working. *Comm. Statist. Simulation Comput.*, **39**, 683-692.

[3] Gertsbakh, I. and Shpungin, Y. (2011). *Network reliability and resilience.* Springer Science & Business Media.

[4] Gertsbakh, I. and Shpungin, Y. (2012), Stochastic models of network survivability. *QTQM*, **9**, 45-58.

[5] Harris, R. (1970), A multivariate definition for increasing hazard rate distribution functions. *The Annals of Mathematical Statistics*, 713-717.