First Seminar on
Reliability Theory and its Applications
27-28, May 2015

On Additive-Multiplicative Hazards Model

Esna-Ashari, M. [1] and Asadi, M. [2]

Department of Statistics, University of Isfahan

Abstract

In survival analysis and reliability theory, a fundamental problem is the study of lifetime properties of a live organism or system. In this regard, there have been considered and studied several models based on different concepts of aging such as hazard rate and mean residual life. In this paper, we consider an additive-multiplicative hazard model (AMHM) and study some of reliability and aging properties of the proposed model. We then specify the bivariate models whose conditionals satisfy AMHM. Several properties of the proposed bivariate model are investigated.

Keywords: Conditionally specified distributions, Bivariate Pareto distribution, Additive hazard, Multiplicative hazard.

1 Introduction

In order to study the lifetime properties of a live organism or system, different approaches have been considered in the literature. In the the context of reliability and survival analysis some of approaches are based on aging concepts such as hazard rate, reversed hazard rate, mean residual life etc. Among the well known models, one can refer to proportional hazards model, proportional mean residual lives model, proportional odds ratio etc, see, for example, Cox (1972), Navarro et al. (2015)). Assume that X denotes the lifetime of a live organism or a device. In the study of aging and stochastic of the system in addition to the main variable (X), to be more realistic one has to consider other observable or unobservable random variable (covariate) which effects the aging characteristics of X such as hazard rate, reversed hazard rate, mean residual life, odds ratio etc. In many applications, the effect of some covariates to the lifetime characteristics are additive while others are multiplicative. There are situation where the effect of covariates on the lifetime

[1]m.esnaashari.stat@gmail.com
[2]m.asadi@sci.ui.ac.ir

characteristic are both the additive and multiplicative (see, for example, Cox (1972), Unnikrishnan and Sankaran (2012)). In additive-multiplicative hazard model (AMHM), one considers a baseline hazard rate, $r(x)$, corresponding to a non-negative random variable and a random variable, Z, representing the covariate with an additive-multiplicative effect on $r(x)$. In other words, in AMHM it is assumed that there are non-negative functions $a(.)$ and $b(.)$ such that the conditional hazard rate $r(x)$, given that $Z = z$, is expressed as:

$$r(x \mid z) = b(z)r(x) + a(z), \tag{1}$$

The different choices for distribution function of Z and flexibility of choosing functions $a(z)$ and $b(z)$ provide situations that one can have a very flexible model to describe variate phenomena in reliability and survival analysis. In particular the model has proportional hazards model as a special case when $a(z) \equiv 0$ and it reduces to the additive hazard model when $b(z) \equiv 1$, which is recently studied by Unnikrishnan and Sankaran (2012). This paper is an investigation on different aging and stochastic properties of the model in (1).

2 Additive−multiplicative hazards model

In this section, we study several properties of AMHM in (1). Note that, if X^* is a random variable satisfying to (1), the survival function of X^* is represented as

$$S^*(x \mid z) = S^{b(z)}(x)\exp(-xa(z)), \quad x, z > 0. \tag{2}$$

The marginal distribution of X^* is given as follows

$$S^*(x) = \int_0^\infty S^{b(z)}(x)\exp(-xa(z))\ g(z)dz. \tag{3}$$

From (6) the joint density of (X^*, Z) is

$$f^*(x, z) = (b(z)r(x) + a(z))S(x)g(z), \quad x, z > 0. \tag{4}$$

It can be easily seen that the following expression is equivalent to (3)

$$r^*(x) = r(x)\frac{\int_0^\infty b(z)\ S^{b(z)}(x)\ e^{-xa(z)}\ g(z)dz}{\int_0^\infty S^{b(z)}(x)\ e^{-xa(z)}\ g(z)dz} + \frac{\int_0^\infty a(z)\ S^{b(z)}(x)\ e^{-xa(z)}\ g(z)dz}{\int_0^\infty S^{b(z)}(x)\ e^{-xa(z)}\ g(z)dz}. \tag{5}$$

Proposition 1. *Suppose that $S(x)$ is a baseline survival function and X^* has the CDF (3) for some functions $a(z)$ and $b(z)$. If both $a(z)$ and $b(z)$ are increasing or both are decreasing, then*

$$r^*(x) \leq r(x)E(b(Z)) + E(a(Z)).$$

In the following we give an example.

Example 1. Assume that Y has generalized gamma distribution with density function

$$g(z) = (\Gamma(\alpha))^{-1}c\lambda^{c\alpha}z^{c\alpha-1}\exp(-(\lambda z)^c), \quad z > 0.$$

Now assume that $a(z) = b(z) = z^c,\ c > 0$. Then, we have

$$S^*(x) = \left(\frac{\lambda^c}{x - \ln S(x) + \lambda^c}\right)^\alpha, \quad x, \lambda, c > 0,$$

or, equivalently, $r^*(x) = \frac{\alpha(r(x)+1)}{x-\ln S(x)+\lambda^c}$.

Theorem 1. *Let $a(z)$ and $b(z)$ be increasing (decreasing) functions and also suppose that $S(x)$ is IFR distribution. Then, the joint density of X^* and Z is RR_2 (TP_2).*

Theorem 2. *Suppose that the functions $a(z)$ and $b(z)$ are both increasing or both decreasing.*

(a) If X^* (X) is IFRA (DFRA) , then so is X (X^*);

(b) If X^* (X) is NBU (NWU), then so is X (X^*);

Theorem 3. *In two AMHM's, suppose that $Z_1 \stackrel{d}{=} Z_2$, where $\stackrel{d}{=}$ stands for equality in distribution. Then*

(i) $X_1 \leq_{st} X_2$ if and only if $X_1^* \leq_{st} X_2^*$.

(ii) If $X_1 \geq_{cx} X_2$,then $X_1^* \geq_{cx} X_2^*$.

Theorem 4. *In two AMHM's, suppose that $X_1 \stackrel{d}{=} X_2$. If $Z_1 \leq_{st} Z_2$ and the functions $a(z)$ and $b(z)$ are both decreasing (increasing), then $X_1^* \leq_{st} X_2^*$ ($\geq_{st}$).*

Constructing bivariate distributions based on conditional distributions is a subject that has been explored by many researchers (see, e.g., Arnold et al. (1993)). In the sequel, we study the bivariate models whose conditional distributions satisfy in AMHM. That is, we are interested in specifying the joint distribution for (X, Y) such that the following conditions are met.

$$P(X > x \mid Y > y) = S_1^{b_1(y)}(x) \exp(-xa_1(y)), \quad x, y > 0, \tag{6}$$

and

$$P(Y > y \mid X > x) = S_2^{b_2(x)}(y) \exp(-ya_2(x)), \quad x, y > 0. \tag{7}$$

Theorem 5. *Suppose that $S_1(x)$ and $S_2(y)$ are baseline reliability functions of X and Y, respectively, and let (X, Y) has common support $(0,\infty) \times (0,\infty)$. Then, the bivariate reliability function with conditionals satisfying (6) and (7) is given by.*

$$S(x, y) = S_1^{\lambda_1 y+\lambda_2}(x) S_2^{\lambda_3 x+\lambda_4}(y) \exp\big(-(\lambda_5 \ln S_1(x) \ln S_2(y) + \lambda_6 x + \lambda_7 y + \phi\lambda_6\lambda_7 xy)\big), \tag{8}$$

for $x, y > 0$, where λ_i, ϕ is nonnegative constants.

In the follwing, we study some reliability and aging properties of the model in (8) in special case when $b_1(y) = b_2(x) = 1$. In other words we are absorbed in specifying the joint distribution for (X, Y) such that the following conditions be satisfied.

$$P(X > x \mid Y > y) = S_1(x) \exp(-xa_1(y)), \quad P(Y > y \mid X > x) = S_2(y) \exp(-ya_2(x)). \tag{9}$$

In this case the the joint reliability function of (X, Y) has the from

$$S(x, y) = S_1(x)S_2(y) \exp(-(\lambda_1 x + \lambda_2 y + \phi\lambda_1\lambda_2 xy)), \tag{10}$$

where λ_1, λ_2 and ϕ are nonnegative constants. The choice of $b_1(y) = b_2(x) = 1$, enable us to have more insight to the properties of the joint distribution of (X, Y). First note

that, from definitions of IFRA and NBU, one can easily conclude that when S_1 and S_2 are univariate IFRA, then (X,Y) is bivariate IFRA and when S_1 and S_2 are both univariate NBU property, then (X,Y) is bivariate NBU property.
From (10), the marginal distributions of X and Y are given respectively by

$$S_X(x) = S(x,0) = S_1(x)\exp(-\lambda_1 x), \quad S_Y(y) = S(0,y) = S_2(y)\exp(-\lambda_2 y)$$

for $x \geq 0$, $y \geq 0$. Hence, the marginal distributions belong to additive hazards model. From these, under the assumption that the derivatives exist, the marginal hazards rates of X and Y are given by

$$r_X(x) = r_1(x) + \lambda_1, \quad r_Y(y) = r_2(y) + \lambda_2. \tag{11}$$

Hence, $r_X(x)(r_Y(y))$ is increasing (decreasing) iff $r_1(x)(r_2(y))$ is increasing (decreasing). The conditional survival functions are also given by

$$P(X > x \mid Y > y) = S_1(x)\exp(-(\lambda_1 + \phi\lambda_1\lambda_2 y)x)$$

and

$$P(Y > y \mid X > x) = S_2(y)\exp(-(\lambda_2 + \phi\lambda_1\lambda_2 x)y)$$

for $x, y \geq 0$. That is , the conditional survival functions belong to additive hazard model, given in (9) , with both $a_1(y)$ and $a_2(x)$increasing functions and

$$a_1(y) = \lambda_1 + \phi\lambda_1\lambda_2 y, \qquad a_2(x) = \lambda_2 + \phi\lambda_1\lambda_2 x, \tag{12}$$

From (10), the bivariate PDF can be expressed as

$$f(x,y) \;=\; \Big((r_1(x) + a_1(y))(r_2(y) + a_2(x)) - \phi\lambda_1\lambda_2\Big)S(x,y), \tag{13}$$

where $r_1(x)$ and $r_2(y)$ are the hazards rate of the baseline distributions S_1 and S_2, respectively.

Theorem 6. *The joint survival function defined in (10) is RR_2 on $(0,\infty) \times (0,\infty)$.*

Theorem 7. *Suppose that the baseline distributions S_1 and S_2 are IFR. Then, the joint PDF of (X,Y) obtained in expression (13) is RR_2 on $(0,\infty) \times (0,\infty)$.*

References

[1] Arnold, B.C., Castillo, E., Sarabia, J.M., 1993. Multivariate distributions with generalized Pareto conditionals. Statistics Probability Letters 17, 361-368.

[2] Cox, D.R., 1972. Regression models and life tables (with discussion). Journal of the Royal Statistical Society Series 34, 187-220.

[3] Navarro, J., Esna-Ashari, M., Asadi, M., Sarabia, J.M., 2015. Bivariate distributions with conditionals satisfying the proportional generalized odds rate model. Metrika DOI 10.1007/s00184-014-0523-7.

[4] Unnikrishnan Nair, N., Sankaran, P.G., 2012. Some results on an additive hazards model. Metrika 75, 389-402.