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Estimation with Non-Homogeneous Sequential K-out-of-N System Lifetimes

Hashempour, M. [1] and Doostparast, M. [2]

Department of Statistics, Ferdowsi University of Mashhad

Abstract

In this paper, sequential k-out-of-n systems with coming non- homogeneous exponential component lifetimes are considered. Point estimates of parameters as well as equal-tail and approximate confidence intervals and Fisher Information are derived on the basis of observed multiply system lifetimes.

Keywords: Bayesian approach, Estimation, Maximum likelihood, Sequential order statistics.

1 Introduction

Kamps [7] introduced the concept of the *sequential order statistics* (SOSs), as an extension of the (usual) order statistics (OSs). SOSs may be used for modelling lifetimes of sequential r-out-of-n systems. Specifically, in (usually) the k-out-of-n system failing a component does not change the lifetimes of the surviving components. Motivated by Cramer and Kamps [3, 4], in practice, the failure of a component may results in a higher load on the remaining components and hence causes the distribution of the surviving components change. In these cases, the system lifetimes may be modelled by SOSs. The mentioned system is called *sequential r-out-of-n system* and the system lifetime is then r-th component failure time, denoted by $X^{\star}_{(r)}$. In the literature, $(X^{\star}_{(1)}, \cdots, X^{\star}_{(n)})$ is called SOSs; See, Kamps [7]. The problem of estimating parameters on the basis of SOS has been considered in the literature. For example, Cramer and Kamps [3] considered the problem of estimating the parameters on the basis of s independent SOSs samples under a

[1]ma.hashempour@stu.um.ac.ir
[2]doustparast@um.ac.ir

proportional hazard rates (PHR) model, defined by $\bar{F}_j(t) = \bar{F}_0^{\alpha_j}(t)$ for $j = 1, \cdots, r$, where the underlying CDF $F_0(t)$ is the exponential distribution, i.e.

$$F_0(x;\sigma) = 1 - \exp\Big\{-\Big(\frac{x}{\sigma}\Big)\Big\}, \quad x > 0, \quad \sigma > 0. \tag{1}$$

In this case, the hazard rate function of the CDF F_j, for $t > 0$ and $j = 1, \cdots, n$, is $h_j(t) = \alpha_j h_0(t)$. See also, Schenk *et al.* [9], Esmailian and Doostparast [5], Beutner and Kamps [1] and references therein.

2 Main results

We assume that $s \geq 2$ independent SOS samples of equal size r from s non-homogeneous populations are available. The data may be represented by $[[x_{ij}]]_{i=1,\cdots,s,j=1,\cdots,r}$ where the i-th row of the matrix $\mathbf{x}$ in (2) denotes the SOS sample coming from the i-th population. The LF of the available data is

$$L(\underline{F_j^{[i]}};\mathbf{x}) = B^s \prod_{i=1}^{s} \left(\prod_{j=1}^{r-1} \left[f_j^{[i]}(x_{ij}) \left(\frac{\bar{F}_j^{[i]}(x_{ij})}{\bar{F}_{j+1}^{[i]}(x_{ij})} \right)^{n-j} \right] f_r^{[i]}(x_{ir}) \bar{F}_r^{[i]}(x_{ir})^{n-r} \right), \tag{2}$$

where $B = \Gamma(n+1)/\Gamma(n-r+1)$, and for $i = 1, \cdots, s$, $j = 1, \cdots, r$. By substituting Equation (1) into Equation(2), under the earlier mentioned PHR model, the LF of the available data reduces to

$$L(\underline{\sigma};\mathbf{x}) = A^s \left(\prod_{j=1}^{r} \alpha_j\right)^s \left(\prod_{i=1}^{s} \frac{1}{\sigma_i}\right)^r \exp\Big\{-\sum_{i=1}^{s}\sum_{j=1}^{r} \Big(\frac{x_{ij} m_j}{\sigma_i}\Big)\Big\}. \tag{3}$$

where $m_j = (n-j+1)\alpha_j - (n-j)\alpha_{j+1}$, for $j = 1, \cdots, r$, with convention $\alpha_{r+1} \equiv 0$. For sake of brevity, we assumed that the proportional parameter vector $\boldsymbol{\alpha}$ are the same among the s sequential r-out-of-n systems. First suppose that the vector parameter $\boldsymbol{\alpha}$ in Equation (3) is known. The solutions of the likelihood equations yields the ML estimate of σ_i $(i = 1, \cdots, s)$ as

$$\hat{\sigma}_i = \frac{\sum_{j=1}^{r} x_{ij} m_j}{r} = \frac{\sum_{j=1}^{r} (n-j+1)\alpha_j D_{ij}}{r}, \tag{4}$$

where $D_{ij} = x_{ij} - x_{i,j-1}$, for $j = 1, \cdots, r$. Cramer and Kamps [4] showed that under the PHR with the one-parameter exponential baseline CDF,

$$T_i = \sum_{j=1}^{r} (n-j+1)\alpha_j D_{ij} \sim \Gamma(r, \sigma_i), \quad i = 1, \cdots, s, \tag{5}$$

where $\Gamma(a,b)$ calls for the gamma distribution. Thus, for $i = 1, \cdots, s$, $\hat{\sigma}_i \sim \Gamma(r, \sigma_i/r)$, and then $E(\hat{\sigma}_i) = \sigma_i$ and $Var(\hat{\sigma}_i) = \sigma_i^2/r$. From Equation (5), $2r\,(\hat{\sigma}_i/\sigma_i) \sim \chi_{2r}$, where χ_ν stands for the chi-square distribution with ν degrees of freedom. So, an equal-tail confidence interval at level $100\gamma\%$ for σ_i $(i = 1, \cdots, s)$ is

$$\left(\frac{2r\hat{\sigma}_i}{\chi_{2r,(1+\gamma)/2}}, \frac{2r\hat{\sigma}_i}{\chi_{2r,(1-\gamma)/2}} \right), \tag{6}$$

where $\chi_{\nu,p}$ calls for the p-th percentile of the χ_ν-distribution. The observed FI, denoted by $[i(\hat{\sigma}_1, \cdots, \hat{\sigma}_s)]$, on the basis of available SOSs data is equal to minus of the Hessian matrix (HM) evaluated at the MLEs of the parameters, i.e. $i(\hat{\sigma}_1, \cdots, \hat{\sigma}_s) = [[(-\partial^2 \log(L)/\partial\sigma_i\partial\sigma_j)_{1\le i,j\le s}]]|_{\sigma_1=\hat{\sigma}_1,\cdots,\sigma_s=\hat{\sigma}_s}$. It is well known that the MLEs have asymptotically normal distribution with mean σ and the variance $[i(\hat{\sigma}_1, \cdots, \hat{\sigma}_s)]^{-1}$. Therefore, an approximate equi-tailed confidence interval for σ_i is

$$\left(\hat{\sigma}_i - z_{\gamma/2}\sqrt{\frac{\hat{\sigma}_i^2}{r}}\ ,\ \hat{\sigma}_i + z_{\gamma/2}\sqrt{\frac{\hat{\sigma}_i^2}{r}}\right), \tag{7}$$

where z_γ stands for the γ-percentile of the standard normal distribution. When the vector parameter $\boldsymbol{\alpha}$ in Equation (3) is unknown, see, e.g., Cramer and Kamps [4] and Hashempour and Doostparast [6].

We here consider the problem of estimating unknown parameters via a strict Bayesian approach. To do this, we assume that $\boldsymbol{\alpha}$ is known and suggest the conjugate prior distributions for the scale parameters $\sigma_i, i = 1, \cdots, s$, i.e.

$$\sigma_i \sim IG(a_i, b_i),\ i = 1, \cdots, s, \tag{8}$$

From Equation (8) and the LF (3), the joint posterior density function of $\sigma_1, \ldots, \sigma_s$ is

$$\pi(\sigma_1, \ldots, \sigma_s \mid \underline{\mathbf{x}}) \propto \prod_{i=1}^{s}\left(\prod_{j=1}^{r}\alpha_j\frac{b_i^{a_i}\sigma_i^{-(a_i+r)-1}}{\Gamma(a_i)}\exp\left\{-\left(\frac{\sum_{j=1}^{r}x_{ij}m_j + b_i}{\sigma_i}\right)\right\}\right). \tag{9}$$

which implies $\sigma_i \mid \underline{\mathbf{x}} \sim IG\left(a_i + r, \sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij} + b_i\right),\ i = 1, \cdots, s.$ As we expected given $\underline{\mathbf{x}}$, the parameter σ_i are independent. Under the squared error loss (SEL) function, the Bayes estimate of the parameter $\sigma_i (i = 1, \cdots, s)$ is

$$\hat{\sigma}_{i,B} \ = \ \frac{\sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij} + b_i}{a_i + r - 1} = \frac{r\hat{\sigma}_i + b_i}{a_i + r - 1}, \tag{10}$$

where $\hat{\sigma}_i$ is the ML estimate of σ_i given by Equation (4). For $i = 1, \cdots, s$, the Bayes estimates (10) is bias, admissible and may be written as a weighted mean of the mean of the prior (8) and the ML estimate (4). The risk function of the Bayes estimates (10) is

$$R(\hat{\sigma}_{i,B}, \sigma_i) = \frac{\sigma_i^2\left(r + (1-a_i)^2\right) + 2b_i\left(1-a_i\right)\sigma_i + b_i^2}{(a_i + r - 1)^2}, \tag{11}$$

which its minimum, as a function of σ_i, occurs at point $b_i(a_i-1)/[(1-a_i)^2+r]$. Notice for $r = n$ and $\alpha_1 = \cdots = \alpha_n = 1$, $\hat{\sigma}_{i,n} = \sum_{j=1}^{n}x_{ij}/n$ and $\hat{\sigma}_{i,B} = (\sum_{j=1}^{n}x_{ij} + b_i)/(a_i + n - 1)$, which are, respectively, the well-known ML and the Bayes estimates of the exponential parameters on the basis of a random sample of size n; See, e.g., Lawless [8].

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