First Seminar on
Reliability Theory and its Applications
27-28, May 2015

Residual Lifetime of Coherent System with Dependent Identically Distributed Components

Kelkinnama, M. [1]

Department of Mathematical Sciences, Isfahan University of Technology

Abstract

In this paper, we study the residual lifetime of coherent system with possibly dependent identically distributed component lifetimes. These results are based on the representation of system reliability function as a distorted function of common reliability function of components.

Keywords: Coherent systems, Residual lifetime, Survival copula, Distorted function, Stochastic orders.

1 Introduction

Consider a coherent system consisting of n possibly dependent components with lifetimes $X_1, ..., X_n$. Suppose that these random variables are identical with common distribution function F and reliability function $\bar{F}$. The dependence among components is represented by the joint reliability function of $(X_1, ..., X_n)$,

$$\bar{F}(x_1, ..., x_n) = Pr(X_1 > x_1, ..., X_n > x_n).$$

Using the Sklar's copula representation, we have

$$\bar{F}(x_1, ..., x_n) = K(\bar{F}(x_1), ..., \bar{F}(x_n)),$$

where, $K(u_1, ..., u_n)$ is reliability copula and $0 < u_i < 1$. In fact, K is an useful tool for modeling dependence between the components. Denote the lifetime of coherent system with $T = \phi(X_1, ..., X_n)$ where ϕ is the structure function of system. Navarro et al. [1] provided an useful representation for system reliability function as a distorted function of

[1] m.kelkinnama@cc.iut.ac.ir

the $\bar{F}$. Let T be the lifetime of a coherent system with identically distributed component lifetimes having the common reliability function $\bar{F}$ and the joint reliability copula K. Then

$$\bar{F}_T(t) = Pr(T > t) = h(\bar{F}(t)),$$

where, h is a function that depends only on copula K and system structure (minimal path sets of system). The function h is a distorted function which is an increasing continuous function from $[0,1]$ to $[0,1]$, $h(0) = 0$ and $h(1) = 1$. The function h is called domination function. If $K(u_1, ..., u_n)$ is exchangeable, i.e. it is permutation invariant, then

$$h(u) = \sum_{i=1}^{n} a_i K(\underbrace{u, ..., u}_{i-times}, \underbrace{1, ..., 1}_{(n-i)-times}),$$

where, $\mathbf{a} = (a_1, ..., a_n)$ is the minimal signature of the system. . In particular, in the i.i.d. case, K is the product copula, hence $h_I(u) = \sum_{i=1}^{n} a_i u^i$ As an example, consider the system $T = \max(X_1, \min(X_2, X_3))$. The minimal path sets are $P_1 = \{1\}$ and $P_2 = \{2, 3\}$.

$$\begin{aligned} Pr(T > t) =& Pr(X_{P_1} > t) + Pr(X_{P_2} > t) - Pr(X_{P_1 \cup P_2} > t) \\ =& \bar{F}(t, 0, 0) + \bar{F}(0, t, t) - \bar{F}(t, t, t) \\ =& K(\bar{F}(t), 1, 1) + K(1, \bar{F}(t), \bar{F}(t)) - K(\bar{F}(t), \bar{F}(t), \bar{F}(t)) = h(\bar{F}(t)) \end{aligned}$$

where, $h(u) = K(u, 1, 1) + K(1, u, u) - K(u, u, u)$. If K is exchangeable then $a = (1, 1, -1)$ is the minimal signature of the system.

In this paper, we study the aging properties and stochastic comparisons of residual lifetimes of coherent systems with dependent identically distributed (DID) component lifetimes. The results derived in this paper can also be applied to coherent systems with exchangeable or i.i.d. components.

2 Main results

For a fixed $t > 0$, the residual lifetime of the coherent system at time t, is denoted by $T^t = [T - t | T > t]$. The reliability function of T^t is

$$\bar{F}_{T^t}(x) = Pr(T^t > x) = \frac{\bar{F}_T(t + x)}{\bar{F}_T(t)} = \frac{h(\bar{F}(t + x))}{h(\bar{F}(t))}.$$

The hazard rate function of T^t can be written as

$$r_{T^t}(x) = r(t + x)\alpha(\bar{F}(t + x)),$$

where, $\alpha(u) = \frac{uh'(u)}{h(u)}$ and r is the hazard rate function of $\bar{F}$.

Theorem 1. *If $\bar{F}$ is IFR and $\alpha(u)$ is a decreasing function of $u \in (0, 1)$, then for the conditional random variable T^t, we have*

(i) $T^t \geq_{hr} T^{t'}$, for $t \leq t'$.

(ii) T^t is IFR for all $t > 0$.

Note that part (i) implies that $T^t \geq_{st} T^{t'}$, i.e. T is IFR.
For the reversed hazard rate function of T^t we have

$$\tilde{r}_{T^t}(x) = \tilde{r}(t+x)\beta(\bar{F}(t+x))\frac{1-h(\bar{F}(t+x))}{h(\bar{F}(t)) - h(\bar{F}(t+x))},$$

where, $\beta(u) = \frac{(1-u)h'(u)}{1-h(u)}$, and $\tilde{r}$ is the reversed hazard rate of $\bar{F}$.

Theorem 2. *Suppose that $\bar{F}$ is $DRHR$, and $\beta(u)$ is increasing in $u \in (0,1)$.*

(i) If $1-h(u)$ is log-concave in u, then $T^t \geq_{rh} T^{t'}$ for $t \leq t'$.

(ii) T^t is $DRHR$ for all $t > 0$.

The Glaser's function (eta function) of T^t can be written as

$$\eta_{T^t}(x) = -\frac{f'_{T^t}(x)}{f_{T^t}(x)} = \eta(t+x) + r(t+x)\gamma(\bar{F}(t+x))$$
$$= \eta(t+x) + \tilde{r}(t+x)\bar{\gamma}(\bar{F}(t+x)),$$

where, $\gamma(u) = \frac{uh''(u)}{h'(u)}, \bar{\gamma}(u) = \frac{(1-u)h''(u)}{h'(u)}$, and η is the eta function of $\bar{F}$.

Theorem 3. *If the common density function, f, is log-concave and there exist $a \in [0,1]$ such that $\gamma(u)$ is non-negative and decreasing in $u \in (0,a)$ and $\bar{\gamma}(u)$ is non-positive and decreasing in $u \in (a,1)$ then*

(i) $T^t \geq_{lr} T^{t'}$, for $t \leq t'$.

(ii) f_{T^t} is log-concave for all $t > 0$.

Navarro et al. [2] showed that, if $\bar{F}$ is $NBU(NWU)$ and $h(u)h(v) \geq (\leq) h(uv)$ for all $0 \leq u, v \leq 1$, then T is $NBU(NWU)$, it means that $T \geq_{st} (\leq_{st}) T^t$ for all $t > 0$. Now, in the next theorem, we give sufficient conditions for some other stochastic orders between T and T^t.

Theorem 4. *(i) If $\bar{F}$ is $IFR(DFR)$ and $\alpha(u) \geq (\leq) 1$, then $T \geq_{hr} (\leq_{hr}) T^t$ for all $t > 0$.*

(ii) If $\bar{F}$ is $DRHR$ and $\beta(u) \leq 1$, then $T \geq_{rh} T^t$ for all $t > 0$.

(iii) If f is log-concave (log-convex) and $\gamma(u) \geq (\leq) 0$ then $T \geq_{lr} (\leq_{lr}) T^t$ for all $t > 0$.

The following theorems provide conditions under which the residual lifetimes of two coherent systems with DID components can be compared.

Theorem 5. *Let $T_1 = \phi_1(X_1, ..., X_n)$ and $T_2 = \phi_2(Y_1, ..., Y_m)$ be the lifetimes of two coherent systems with DID components having common reliability function $\bar{F}$. Let h_1 and h_2 be their respective domination functions. Then, we have the following properties for all $t > 0$.*

(i) $T_1^t \leq_{st} (\geq_{st}) T_2^t$ for all $\bar{F}$ if and only if $\frac{h_2(u)}{h_1(u)}$ is decreasing (increasing) in $u \in (0,1)$.

(ii) $T_1^t \leq_{hr} (\geq_{hr}) T_2^t$ for all $\bar{F}$ if and only if $\frac{h_2(u)}{h_1(u)}$ is decreasing (increasing) in $u \in (0,1)$.

(iii) $T_1^t \leq_{rhr} (\geq_{rhr}) T_2^t$ *for all* $\bar{F}$ *if and only if* $\frac{h_2(q)-h_2(u)}{h_1(q)-h_1(u)}$ *is decreasing (increasing) in* $u \in (0, q)$.

(iv) $T_1^t \leq_{lr} (\geq_{lr}) T_2^t$ *for all* $\bar{F}$ *if and only if* $\frac{h_2'(u)}{h_1'(u)}$ *is decreasing (increasing) in* $u \in (0, 1)$.

Theorem 6. *Let* $T_1 = \phi(X_1, ..., X_n)$ *and* $T_2 = \phi(Y_1, ..., Y_n)$ *be the lifetimes of two coherent systems with the same structure and with DID component lifetimes having the same copula and common absolutely continuous reliability functions* $\bar{F}$ *and* $\bar{G}$*, respectively. Let* h *be the domination function and assume that it is twice differentiable. Then, we have the following properties for all* $t > 0$.

(i) If $X_1 \leq_{st} Y_1$ *and* $h(u)$ *is log-concave in* u*, then* $T_1^t \leq_{st} T_2^t$.

(ii) If $X_1 \leq_{hr} Y_1$ *and* $\frac{uh'(u)}{h(u)}$ *is decreasing in* u*, then* $T_1^t \leq_{hr} T_2^t$.

(iii) If $X_1 \leq_{rhr} Y_1$, $\frac{(1-u)h'(u)}{1-h(u)}$ *is increasing in* u*, and* $\frac{1-h(u_1)}{h(q_1)-h(u_1)} \leq \frac{1-h(u_2)}{h(q_2)-h(u_2)}$ *for* $u_1 \leq u_2$, $u_1 \in (0, q_1), u_2 \in (0, q_2), q_1 \leq q_2$*, then* $T_1^t \leq_{rhr} T_2^t$.

(iv) If $X_1 \leq_{lr} Y_1$ *and* $\frac{uh''(u)}{h'(u)}$ *is non-negative and decreasing in* u*, then* $T_1^t \leq_{lr} T_2^t$.

References

[1] Navarro, J., Aguila, Y., Sordo, M.A. and Suarez-Llorens, A. (2012), Stochastic ordering properties for systems with dependent identically distributed components. *Appl. Stoch. Models Bus. Ind.*, **29**, 264–278

[2] Navarro, J., Aguila, Y., Sordo, M.A. and Suarez-Llorens, A. (2013), Preservation of reliability classes under the formation of coherent systems. *Appl. Stoch. Models Bus. Ind.*, **30**, 444–454.