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Stochastic Comparicons of Generalized Residual Entropy of Order Statistics

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Abstract

In modeling of biological and engineering systems often requires use of concepts of information theory, and in particular of entropy. The concept of residual entropy is applicable to a system which has survived for some units of time. In this paper, we propose a generalized residual entropy based on order statistics and obtain some results on the stochastic comparisons of it.

Keywords: Generalized residual entropy, Hazard rate function, Order statistics, Stochastic comparisons.

1 Introduction

Throughout this paper, X and Y will denote two random variables and the distribution function, density function and hazard rate function of X be denoted by $F(t)$, $f(t)$ and $\lambda_F(t)$ and those of Y be denoted by $G(t)$, $g(t)$ and $\lambda_G(t)$, respectively. We will be particularly interested in X_t, the remaining lifetime of a unit of age $t \geq 0$. That is, $X_t \overset{d}{=} X - t|X > t$ where $\overset{d}{=}$ stands for equality in distribution. For each $t \geq 0$, the probability distribution of X_t is absolutely continuous with distribution function $F_t(x) = P(X - t \leq x|X > t) = \frac{[F(x+t)-F(t)]}{\bar{F}(t)}, x > 0$, survival function $\bar{F}_t(x) = 1 - F_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}, x > 0$, and probability density function $f_t(x) = \frac{f(x+t)}{\bar{F}(t)}, x > 0$.
As is well known, an early definition of a measure of the entropy has been introduced by Shannon (1948). Further, Nanda and Paul (2006) introduced a measure of residual

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entropy over (t,∞) based on the Tsallis entropy that is a generalisation of order β of the Shannon entropy (Tsallis, 1988), given by

$$H^{\beta}(X;t)=\frac{1}{\beta-1}\left[1-\frac{\int_t^{+\infty}f^{\beta}(x)dx}{\bar{F}^{\beta}(t)}\right],\ \beta\neq 1,\ \beta>0. \tag{1}$$

Obviously $H^{\beta}(X;0)$ results in Tsallis entropy and $\beta\longrightarrow 1$ gives Shannon entropy. In this paper, we extend this generalized residual entropy based on order statistics and we derive some stochastic comparisons based on the generalized residual entropy and order statistics version of it.

2 Generalized residual entropy of order statistics

Suppose that $X_1,X_2,\ldots,X_n$ are independent and identically distributed observations from cdf $F(t)$ and pdf $f(t)$. The order statistics of the sample is defined by the arrangement of $X_1,X_2,\ldots,X_n$ from the smallest to the largest, denoted as $X_{1:n}\leq X_{2:n}\leq\ldots\leq X_{n:n}$. Generalized residual entropy associated with the i^{th} order statistics $X_{i:n}$ is given by

$$H^{\beta}(X_{i:n};t)=\frac{1}{\beta-1}[1-\frac{\int_t^{\infty}f_{i:n}^{\beta}(x)dx}{\overline{F}_{i:n}^{\beta}(t)}],\quad \beta\neq 1,\beta>0, \tag{2}$$

where $f_{i:n}(x)$ and $\overline{F}_{i:n}(x)$ are the density function and survival function of $X_{i:n}$, respectively (see Davide and Nagaraja, 2003).

Now, using probability integral transformation $U=F(X)$, where U is standard uniform distribution (2) can be expressed as

$$H^{\beta}(X_{i:n};t)=\frac{1}{\beta-1}\left[1-\frac{\overline{B}_{F(t)}(\beta(i-1)+1,\beta(n-i)+1)E[f^{\beta-1}(F^{-1}(Y_i))}{\overline{B}^{\beta}_{F(t)}(i,n-i+1)}\right],$$

where $Y_i\sim\overline{B}_{F(t)}(\beta(i-1)+1,\beta(n-i)+1)$.

3 Stochastic comparisons

Notation: a. The definitions of stochastic comparisons used in this section is available in Shaked and Shanthikumar (1994).
b. The proof theorems stated in brief.

Definition 1. *A random variable X is said to be smaller than Y in Tsallis entropy ordering (denoted by $X\overset{GRE}{\leqslant}Y$) if $H^{\beta}(X;t)\leqslant H^{\beta}(Y;t)$ for all $t>0$.*

It is well known that $X\overset{LR}{\leqslant}Y\Rightarrow X\overset{HR}{\leqslant}Y\Rightarrow X\overset{ST}{\leqslant}Y$ and $X\overset{DIDP}{\leqslant}Y\Rightarrow X\overset{ST}{\leqslant}Y$ and $X\overset{LR}{\leqslant}Y\Rightarrow X\overset{ST}{\leqslant}Y$ (Bickel and Lehmann, 1976; and Shaked and Shanthikumar, 1994). We first prove the following preliminary results for generalized residual entropy.

Theorem 1. *Let X and Y be two random variables, then $X\overset{DISP}{\leq}Y$ implies $X\overset{GRE}{\leq}Y$.*

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Proof. From (1), we have

$$H^{\beta}(X;t)=\frac{1}{\beta-1}\left[1-\frac{B(1,\beta)}{\overline{F}^{\beta}(t)}E(\lambda_F^{\beta-1}(F^{-1}(W)))\right],$$

where $W\sim B(1,\beta)$. we also note that $X\overset{DISP}{\leq}Y$ if and only if $\lambda_G(G^{-1}(u))\leq\lambda_F(F^{-1}(u))$ for all $u\in(0,1)$ (see Shaked and Shanthikumar, 1994). First, we assume that $\beta>1$, on the other hand from Remark 3, $X\overset{DISP}{\leq}Y$ implies that $X\overset{ST}{\leq}Y$. Hence, we find

$$\begin{aligned}H^{\beta}(X;t)-H^{\beta}(Y;t)&\leq\frac{B(1,\beta)}{\beta-1}\left[\frac{1}{\overline{G}^{\beta}(t)}-\frac{1}{\overline{F}^{\beta}(t)}\right]\cdot E(\lambda_F^{\beta-1}(F^{-1}(W)))\\&\leq 0.\end{aligned}$$

Thus, we obtain $X\overset{GRE}{\leq}Y$. For $0<\beta<1$ the proof is similar. □

Theorem 2. *Let X and Y be two random variables, at least one of them is DFR. Then $X\overset{HR}{\leq}Y$ implies $X\overset{GRE}{\leq}Y$.*

Proof. First, we assume that $0<\beta<1$ and X is DFR. Since $X\overset{HR}{\leq}Y$ implies that $X_t\overset{ST}{\leq}Y_t$ (see Shaked and Shanthikumar, 1994) and from (1), we have

$$\begin{aligned}H^{\beta}(X;t)&=\frac{1}{\beta-1}\left[1-E_{f_{X_t,\beta}}(\lambda_F^{\beta-1}(t+X_t))\right]\\&\leq\frac{1}{\beta-1}\left[1-E_{g_{Y_t,\beta}}(\lambda_F^{\beta-1}(t+Y_t))\right]\\&\leq\frac{1}{\beta-1}\left[1-E_{g_{Y_t,\beta}}(\lambda_G^{\beta-1}(t+Y_t))\right]=H^{\beta}(Y;t),\end{aligned}$$

where $f_{X_t,\beta}=\frac{-d\bar{F}_t^{\beta}(x)}{dx}$. For $\beta>1$ the proof is similar. □

Now, by the fact that, $X\overset{DISP}{\leqslant}Y$ implies that $X_{i:n}\overset{DISP}{\leqslant}Y_{i:n}$ (Shaked and Shanthikumar, 1994) and by Theorem 1, we have the following result. Let X and Y be two random variables. Then $X\overset{DISP}{\leqslant}Y$ implies $X_{i:n}\overset{GRE}{\leqslant}Y_{i:n}$.

Theorem 3. *Suppose X has a DFR distribution. Then $X_{i:n}\overset{GRE}{\leqslant}X_{j:n},\ i<j$.*

Proof. Using the result of Chan et al. (1991), we have $X_{i:n}\overset{LR}{\leqslant}X_{j:n}$. By Remark 3, this implies that $X_{i:n}\overset{HR}{\leqslant}X_{j:n}$. Since X has a DFR distribution , $X_{i:n}$ has a DFR distribution, (see Takahasi, 1988). So, by using Theorem 2 , we can conclude that $X_{i:n}\overset{GRE}{\leqslant}X_{j:n}$. □

Theorem 4. *Let $X_1,X_2,\ldots,X_{n+1}$ be iid random variables with distribution function $F(t)$. Suppose X has a DFR distribution. Then, $X_{1:n+1}\overset{GRE}{\leqslant}X_{1:n}$ and $X_{n:n}\overset{GRE}{\leqslant}X_{n+1:n+1}$.*

Proof. We use the fact that $X_{j:m}\overset{LR}{\leqslant}X_{i:n}$ whenever $j\leqslant i$ and $m-j\geqslant n-i$ (Shaked and Shanthikumar, 2007), and the method used in the proof of Theorem 3. □

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