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On the Dynamic Proportional Odds Model

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Abstract

The proportional odds model plays an important role in analyzing survival data.This note develops the definition of dynamic proportional odds (DPO) model and its properties including some results on stochastic comparisons. One application of DPO model is considered as Marshall and Olkin family of distribution in dynamic situation.

Keywords: Proportional odd model , Survival analysis, Marshall and Olkin distribution.

1 Introduction

The proportional odds model which was introduced by Bennett (1983) is appropriate to analyze data in survival analysis. In survival studies, heterogeneity in the population of lifetimes is usually represented by covariates. The main objective in such studies is to understand and exploit the relationship between lifetime and covariates. Parametric and semi parametric regression models are used to analyze such lifetime data. Commonly used semi parametric regression model is Coxs (1972) proportional hazards model. In practical situations, it is not uncommon for the hazard func- ions obtained for two groups to converge with time. In the situations where the data exhibit non-proportional hazards, proportional odds model can be employed. For more details, one could refer to Kirmani and Gupta (2001) and Wang et al. (2003). The proportional odds (PO) frailty model is defined by Marshall and Olkin (1997). Also, see Marshall and Olkin (2007). Its extensions andmodications have been studied by various authors including Gupta and Peng (2009). Since different distributions of frailty give rise to different population-level distribution for analyzing survival data, it is appropriate to investigate how the comparative effect of two frailties translates into the comparative effect on the resulting survival distribution.

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Frequently, in reliability and survival analysis the problem of interest is the lifetime beyond an age t. For example, when a system is working at time t, one is interested in obtaining the reliability of the system beyond$t$. In such case, the random variable of interest, for computing the reliability of the system, is the residual random variable $X_t = X - t|X > t$ with survival function

$$\bar{F}_t(x) = \begin{cases} \frac{\bar{F}(x)}{\bar{F}(t)} & \text{if } x \epsilon S_t, \\ 1 & \text{o.w} \end{cases}$$

where $\bar{F}$ denotes the survival function of X and $S_t = \{x; x > t\}$

Here we briefly recall the definition of proportional odds and proportional odds frailty model .See Bennett (1983) , Kirmani and Gupta (2001),Marshall and Olkin (1997) and the referencess therein for more details about these models. Let X_0 and X be two nonnegative random variables ,with cumulative distribution functions $F_0(x)$ and $F(x)$ and survival functions $\bar{F_0}(x)$ and $F\bar{(}x)$, respectively.The odds functions of X_0 and X are given by

$$\phi_0(x) = \frac{\bar{F_0}(x)}{F_0(x)} \tag{1}$$

and

$$\phi(x) = \frac{\bar{F}(x)}{F(x)} \tag{2}$$

for $x \geq 0$,respectively.We say that X_0 and X satisfy the proporttional odds model with positive proportional constant α if

$$\phi(x) = \alpha\phi_0(x), x \geq 0, \tag{3}$$

where $\phi(x)$ and $\phi_0(x)$ are the population odds function and the baseline , respectively.

Recently, Marshall and Olkin introduced a family of distributions by adding a new parameter to a survival function. Suppose that the $F(. \mid \gamma)$ is defined in terms of the underlying distribution F by the formula

$$\frac{\bar{F}(x \mid \gamma)}{F(x \mid \gamma)} = \gamma\frac{\bar{F}(x)}{F(x)} \tag{4}$$

The family $\{F(. \mid \gamma), \gamma > 0\}$ is said proportional odds frality model.

2 Dynamic proportional odds model

In this section we will give the defenition and some results about DPO and PO frailty models.

Definition 1. *Let X_0 and X be two nonnegative random variables with cumulative distribution functions $F_0(x)$ and $F(x)$ and survival functions $\bar{F_0}(x)$ and $F\bar{(}x)$, respectively. The odds functions of X_0 and residual odds function X are given by*

$$\phi_0(x) = \frac{\bar{F_0}(x)}{F_0(x)} \tag{5}$$

and

$$\phi_t(x) = \frac{\bar{F_t}(x)}{F_t(x)} \tag{6}$$

for $x \geq 0$*,respectively.We say that* X_0 *and* X *satisfy the dynamic proportional odds (DPO)model with positive continuous proportional function* $\alpha(x,t)$ *if*

$$\phi_t(x) = \alpha(x,t)\phi_0(x) \tag{7}$$

where $\phi_t(x)$ *and* $\phi_0(x)$ *are the dynamic population odds function and the baseline one,respectively.*

Example 1. *By appling residul life time distributions in PO model then DPO has the following structure.*

$$\phi_t(x) = \alpha\frac{F_0(x)}{F_0(x) - F_0(t)}\phi_0(x) \tag{8}$$

It is easily seen that PO model is a special case of DPO model as $t \to 0$*.*

Now we develop some properties on stochastic comparisons of the dynamic proportional odds model.

Theorem 1. *Suppose (2.3)holds,*

1. *if* $0 < \alpha(x,t) \leq 1$ *, then* $X_t \leq_{lr} X_0$
2. *if* $\alpha(x,t) \geq 1$ *,then* $X_0 \leq_{lr} X_t$

where Lr denotes likelihood ratio order , X_0 and X_t are the baseline variable and the residual of population variable respectively.

Let X_0 and X be two nonnegative random variables that satisfy the dynamic proportional odds (DPO)model

$$\phi_t(x) = \alpha(x,t)\phi_0(x) \tag{9}$$

and Y_0 and Y be two nonnegative random variables that satisfy the dynamic proportional odds (DPO)model

$$\psi_t(x) = \beta(x,t)\psi_0(x) \tag{10}$$

Theorem 2. *Suppose (2.4) and (2.5) are satisfied,*
if $\alpha \leq \beta$ *,* $X_0 \leq_{st} Y_0$ *and* $G_0(x) - G_0(t) \leq F_0(x) - F_0(t)$ *then*

$$X \leq_{st} Y \tag{11}$$

Definition 2. *Suppose that the residul life time* $F_t(. \mid \gamma)$ *is defined in terms of the underlying residual distribution* F_t *by the formula*

$$\frac{\bar{F}_t(x \mid \gamma)}{F_t(x \mid \gamma)} = \alpha\frac{\bar{F}_t(x)}{F_t(x)}. \tag{12}$$

then the family $\{F_t(. \mid \gamma), \gamma > 0\}$ *is said dynamic proportional odds frality model.*

Example 2. *Suppose underlying distribution has exponnetial distribution then by (2.8) we have*

$$\bar{F}_t(x|\lambda) = \frac{\alpha e^{-\lambda x}}{e^{-\lambda t} - \bar{\alpha}e^{-\lambda x}} \tag{13}$$

Theorem 3. *If* F *has a density* f *and hazard rate* r*. then for* $\gamma > 0$*, the hazard rate distribution* $F_t(x \mid \gamma)$ *is given by*

$$r_t(x|\gamma) = \frac{r(x)\bar{F}(t)}{\bar{F}(t) - \bar{\gamma}\bar{F}(x)} \tag{14}$$

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