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On Properties of Log-Odds Function

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Abstract

In this paper, first we introduce the log-odds (LO) and log-odds ratio (LOR) functions and their relations with reliability concepts such as hazard and reversed hazard rate. Then, we proposed a new measure of skewness based on LO function in discrete and continuous lifetime distributions and compare it with Pearson's moment coefficient of skewness and also Groeneveld-Meeden measure of skewness via some examples. Also some results due to bivariate log-odds are discussed.

Keywords: Log-odds rate, Hazard rate, Reversed hazard rate, Second hazard rate, Second reversed rate of failure.

1 Introduction

Zimmer et al. [6] and Wang et al. [4, 5] introduced a new model for continuous time to failure based on the log-odds rate (LOR) which is comparable to the model based on the failure rate. Also Khorashadizadeh et al. [2] defined the discrete log-odds rate and have obtained some characterization results for discrete lifetime distributions.

Suppose that X be a non-negative continuous random variable with probability density function (pdf) $f_X(x)$, cumulative density function (cdf) $F_X(x) = P(X \leq x)$ and reliability function $R_X(x) = P(X > x)$, then the LOR function is defined by $\text{LOR}_X(x) = \frac{\partial}{\partial x}\text{LO}_X(x)$, where $\text{LO}_X(x) = \ln \frac{F_X(x)}{R_X(x)}$ is the log-odds function. Hence,

$$\text{LOR}_X(x) = \frac{f_X(x)}{F_X(x)R_X(x)} = \frac{h_X(x)}{F_X(x)} = \frac{r_X(x)}{R_X(x)} = h_X(x) + r_X(x),$$

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where $h_X(x)=\frac{f_X(x)}{R_X(x)}$ is the hazard rate and $r_X(x)=\frac{f_X(x)}{F_X(x)}$ is the reversed hazard rate. The log-odds rate function characterizes the distribution uniquely [4].

Let T be a non-negative discrete random variable with probability mass function (pmf) $p_T(t)$, cdf $F_T(t)=P(T\le t)$ and reliability function $R_T(t)=1-F(t)=P(T>t)$. Then the LO function is defined by $\mathrm{LO}_T(t)=\ln\frac{F_T(t)}{R_T(t)}$. Khorashadizadeh et al. [2] have shown that,

$$\mathrm{LOR}^*_T(t)=\mathrm{LO}_T(t)-\mathrm{LO}_T(t-1)=r^*_T(t)+h^*_T(t),$$

where $r^*_T(t)=\ln\frac{F_T(t)}{F_T(t-1)}$ is the second reversed rate of failure and $h^*_T(t)=-\ln\frac{R_T(t)}{R_T(t-1)}$ is the second rate of failure. The $\mathrm{LOR}^*_T(t)$ function also characterizes the distribution function uniquely [2].

By changing the variables, $Y=\ln X$, $(X=e^Y)$, in continuous case the log-odds rate in terms of $\ln x$, we have $\mathrm{LOR}_Y(y)=\frac{h_Y(y)}{F_Y(y)}=e^y\frac{h_X(e^y)}{F_X(e^y)}$ for $\quad y\ge 0$. Wang et al. [4, 5] proved the following relation, under mild condition, which is usually satisfied in reliability practice,

$$\text{ILOR}\quad\text{in}\quad x\Rightarrow\text{IFR}\Rightarrow\text{ILOR}\quad\text{in}\quad \ln x.$$

The class of log-odds rate in terms of $\ln x$ is more interesting than log-odds rate in terms of x, because the class of LOR in terms of $\ln t$ is weaker than the class of IFR.

Also, for discrete case in terms of $K=\ln T,(T=e^K)$ it has been shown that [2], $\mathrm{LOR}^*_K(k)=\sum_{i=1}^t(r^*_T(i)+h^*_T(i))-\ln\frac{F_T(te^{-1})}{R_T(te^{-1})}+a$, where $a=\ln\frac{p_T(0)}{1-p_T(0)}$.

In general for continuous lifetime distribution we have:

- F has constant LOR in x $(\ln x)$ if and only if F has a logistic (log logistic) distribution.
- If F has a Burr XII distribution with parameters α and β, then, for $\beta=1$, it reduces to log logistic distribution and has constant LOR in $\ln x$, and for $\beta>1(\beta<1)$, it is ILOR (DLOR) in $\ln x$.

Also, in discrete lifetime distribution we have,

- F is ILOR in terms of $t\,(k=\ln t)$ if and only if the LO function is convex with respect to (w.r.t), $t\,(k=\ln t)$. Also, for dual class DLOR it is true for concave function.
- If T has a discrete standard logistic distribution, then $LO_T(t)=t+1$ and by simple transformation the discrete truncated logistics distribution has constant LOR in t.
- If T has a discrete Burr XII distribution with parameters α and θ, then in terms of $\ln t$, F is ILOR for $\theta<e^{-1}$, constant for $\theta=e^{-1}$, and DLOR for $\theta>e^{-1}$.

2 Measure of skewness based on LO

If we define $SM=\int LO(x)dx$, in continuous case and $SM=\sum LO(t)$ in discrete distributions, then these measures may be measure of skewness.

Theorem 1. *Let X be a continuous random variable with cdf, $F(x)$ and log-odds function, $LO(x)=\ln\frac{F(x)}{1-F(x)}$, then if SM be finite such that, $SM=\int_{-\infty}^{\infty}LO(x)dx$, we have, $F(x)$ is symmetric (positive or negative skewed) if and only if $SM=(\ge$ or $\le)0$.*

Proof:
Suppose X has a symmetric distribution, then we have, $F(x) = \overline{F}(2M - x)$, where M is its median (or mean) and therefore $LO(x) = -LO(2M - x)$. Thus, $SM = \int_{-\infty}^{M} LO(x)dx + \int_{M}^{+\infty} LO(x)dx$, so, using the transformation $x = 2M - t$, we have $SM = 0$. Also, when F is positive (negative) skewed, $F(x) > (<)\overline{F}(2M - x)$ and therefore $LO(x) > (<) - LO(2M - x)$, so the "only if" part is proved. The "if" can be proved on contrary.□

Similar results of Theorem 1 can be proved for discrete distribution, using $SM = \sum_{-\infty}^{\infty} LO(t)$. It should be noted that, since SM is just related to cdf, it estimating is more easier than other measures of skewness like Pearson's moment coefficient of skewness [3], $\gamma_1 = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$ and also Groeneveld-Meeden measure of skewness [1] $\gamma_2 = \frac{(\mu - M)}{E|X-M|}$, where M is median.

3 Bivariate case

Let $LO_1(x)$ and $LO_2(y)$ denote the marginal log odds functions of $F_1(x)$ and $F_2(y)$ respectively. The bivariate log odds function can be defined as,

$$LO(x, y) = \ln\left(\frac{F(x, y)}{1 - F(x, y)}\right).$$

We obtained the following properties for $LO(x, y)$:

- The joint distribution can be determined uniquely by,

$$F(x, y) = \frac{1}{1 + e^{-LO(x,y)}}.$$

- If X and Y be two independent random variables, then we have,

$$LO(x, y) = LO_1(x) + LO_2(y) - \ln\left(1 + e^{LO_1(x)} + e^{LO_2(y)}\right).$$

In similar way of Theorem 1, we can proved the following theorem.

Theorem 2. *The bivariate distribution of the random variable (X, Y), is radial symmetric if and only if,*

$$BSM = \int_R \int_R LO^*(x, y)dxdy = 0,$$

where $LO^*(x, y) = \ln \frac{F(x,y)}{\overline{F}(x,y)}$.

Future of the Work

Studying the estimation of the skewness based on data and also a similar definition of skewness and symmetric in bivariate cases are the future of the work.

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