



## On Properties of Log-Odds Function

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### Abstract

In this paper, first we introduce the log-odds (LO) and log-odds ratio (LOR) functions and their relations with reliability concepts such as hazard and reversed hazard rate. Then, we proposed a new measure of skewness based on LO function in discrete and continuous lifetime distributions and compare it with Pearson's moment coefficient of skewness and also Groeneveld-Meeden measure of skewness via some examples. Also some results due to bivariate log-odds are discussed.

**Keywords:** Log-odds rate, Hazard rate, Reversed hazard rate, Second hazard rate, Second reversed rate of failure.

## 1 Introduction

Zimmer et al. [6] and Wang et al. [4, 5] introduced a new model for continuous time to failure based on the log-odds rate (LOR) which is comparable to the model based on the failure rate. Also Khorashadizadeh et al. [2] defined the discrete log-odds rate and have obtained some characterization results for discrete lifetime distributions.

Suppose that  $X$  be a non-negative continuous random variable with probability density function (pdf)  $f_X(x)$ , cumulative density function (cdf)  $F_X(x) = P(X \leq x)$  and reliability function  $R_X(x) = P(X > x)$ , then the LOR function is defined by  $\text{LOR}_X(x) = \frac{\partial}{\partial x} \text{LO}_X(x)$ , where  $\text{LO}_X(x) = \ln \frac{F_X(x)}{R_X(x)}$  is the log-odds function. Hence,

$$\text{LOR}_X(x) = \frac{f_X(x)}{F_X(x)R_X(x)} = \frac{h_X(x)}{F_X(x)} = \frac{r_X(x)}{R_X(x)} = h_X(x) + r_X(x),$$

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where  $h_X(x) = \frac{f_X(x)}{R_X(x)}$  is the hazard rate and  $r_X(x) = \frac{f_X(x)}{F_X(x)}$  is the reversed hazard rate. The log-odds rate function characterizes the distribution uniquely [4].

Let  $T$  be a non-negative discrete random variable with probability mass function (pmf)  $p_T(t)$ , cdf  $F_T(t) = P(T \leq t)$  and reliability function  $R_T(t) = 1 - F(t) = P(T > t)$ . Then the LO function is defined by  $LO_T(t) = \ln \frac{F_T(t)}{R_T(t)}$ . Khorashadizadeh et al. [2] have shown that,

$$LOR_T^*(t) = LO_T(t) - LO_T(t-1) = r_T^*(t) + h_T^*(t),$$

where  $r_T^*(t) = \ln \frac{F_T(t)}{F_T(t-1)}$  is the second reversed rate of failure and  $h_T^*(t) = -\ln \frac{R_T(t)}{R_T(t-1)}$  is the second rate of failure. The  $LOR_T^*(t)$  function also characterizes the distribution function uniquely [2].

By changing the variables,  $Y = \ln X$ , ( $X = e^Y$ ), in continuous case the log-odds rate in terms of  $\ln x$ , we have  $LOR_Y(y) = \frac{h_Y(y)}{F_Y(y)} = e^y \frac{h_X(e^y)}{F_X(e^y)}$  for  $y \geq 0$ . Wang et al. [4, 5] proved the following relation, under mild condition, which is usually satisfied in reliability practice,

$$ILOR \text{ in } x \Rightarrow IFR \Rightarrow ILOR \text{ in } \ln x.$$

The class of log-odds rate in terms of  $\ln x$  is more interesting than log-odds rate in terms of  $x$ , because the class of LOR in terms of  $\ln t$  is weaker than the class of IFR.

Also, for discrete case in terms of  $K = \ln T$ , ( $T = e^K$ ) it has been shown that [2],  $LOR_K^*(k) = \sum_{i=1}^t (r_T^*(i) + h_T^*(i)) - \ln \frac{F_T(te^{-1})}{R_T(te^{-1})} + a$ , where  $a = \ln \frac{p_T(0)}{1-p_T(0)}$ .

In general for continuous lifetime distribution we have:

- $F$  has constant LOR in  $x$  ( $\ln x$ ) if and only if  $F$  has a logistic (log logistic) distribution.
- If  $F$  has a Burr XII distribution with parameters  $\alpha$  and  $\beta$ , then, for  $\beta = 1$ , it reduces to log logistic distribution and has constant LOR in  $\ln x$ , and for  $\beta > 1$  ( $\beta < 1$ ), it is ILOR (DLOR) in  $\ln x$ .

Also, in discrete lifetime distribution we have,

- $F$  is ILOR in terms of  $t$  ( $k = \ln t$ ) if and only if the LO function is convex with respect to (w.r.t),  $t$  ( $k = \ln t$ ). Also, for dual class DLOR it is true for concave function.
- If  $T$  has a discrete standard logistic distribution, then  $LO_T(t) = t + 1$  and by simple transformation the discrete truncated logistics distribution has constant LOR in  $t$ .
- If  $T$  has a discrete Burr XII distribution with parameters  $\alpha$  and  $\theta$ , then in terms of  $\ln t$ ,  $F$  is ILOR for  $\theta < e^{-1}$ , constant for  $\theta = e^{-1}$ , and DLOR for  $\theta > e^{-1}$ .

## 2 Measure of skewness based on LO

If we define  $SM = \int LO(x)dx$ , in continuous case and  $SM = \sum LO(t)$  in discrete distributions, then these measures may be measure of skewness.

**Theorem 1.** Let  $X$  be a continuous random variable with cdf,  $F(x)$  and log-odds function,  $LO(x) = \ln \frac{F(x)}{1-F(x)}$ , then if  $SM$  be finite such that,  $SM = \int_{-\infty}^{\infty} LO(x)dx$ , we have,  $F(x)$  is symmetric (positive or negative skewed) if and only if  $SM = (\geq \text{ or } \leq) 0$ .

**Proof:**

Suppose  $X$  has a symmetric distribution, then we have,  $F(x) = \bar{F}(2M - x)$ , where  $M$  is its median (or mean) and therefore  $LO(x) = -LO(2M - x)$ . Thus,  $SM = \int_{-\infty}^M LO(x)dx + \int_M^{+\infty} LO(x)dx$ , so, using the transformation  $x = 2M - t$ , we have  $SM = 0$ . Also, when  $F$  is positive (negative) skewed,  $F(x) > (<) \bar{F}(2M - x)$  and therefore  $LO(x) > (<) -LO(2M - x)$ , so the "only if" part is proved. The "if" can be proved on contrary.  $\square$

Similar results of Theorem 1 can be proved for discrete distribution, using  $SM = \sum_{-\infty}^{\infty} LO(t)$ . It should be noted that, since  $SM$  is just related to cdf, it estimating is more easier than other measures of skewness like Pearson's moment coefficient of skewness [3],  $\gamma_1 = E \left[ \left( \frac{X-\mu}{\sigma} \right)^3 \right]$  and also Groeneveld-Meeden measure of skewness [1]  $\gamma_2 = \frac{(\mu-M)}{E|X-M|}$ , where  $M$  is median.

### 3 Bivariate case

Let  $LO_1(x)$  and  $LO_2(y)$  denote the marginal log odds functions of  $F_1(x)$  and  $F_2(y)$  respectively. The bivariate log odds function can be defined as,

$$LO(x, y) = \ln \left( \frac{F(x, y)}{1 - F(x, y)} \right).$$

We obtained the following properties for  $LO(x, y)$ :

- The joint distribution can be determined uniquely by,

$$F(x, y) = \frac{1}{1 + e^{-LO(x, y)}}.$$

- If  $X$  and  $Y$  be two independent random variables, then we have,

$$LO(x, y) = LO_1(x) + LO_2(y) - \ln \left( 1 + e^{LO_1(x)} + e^{LO_2(y)} \right).$$

In similar way of Theorem 1, we can proved the following theorem.

**Theorem 2.** *The bivariate distribution of the random variable  $(X, Y)$ , is radial symmetric if and only if,*

$$BSM = \int_R \int_R LO^*(x, y) dx dy = 0,$$

where  $LO^*(x, y) = \ln \frac{F(x, y)}{\bar{F}(x, y)}$ .

### Future of the Work

Studying the estimation of the skewness based on data and also a similar definition of skewness and symmetric in bivariate cases are the future of the work.

# References

- [1] Groeneveld, R.A. and Meeden, G. (1984), Measuring Skewness and Kurtosis, *Journal of the Royal Statistical Society, Series D (The Statistician)*, **33**, 391-399.
- [2] Khorashadizadeh, M., Rezaei Roknabadi, A.H. and Mohtashami Borzadaran, G.R. (2013), Characterization of life distributions using Log-odds rate in discrete aging, *Communications in Statistics - Theory and Methods*, **42**, 1, 76-87.
- [3] Pearson, K. (1895), Contributions to the Mathematical Theory of Evolution, II: Skew Variation in Homogeneous Material, *Transactions of the Royal Philosophical Society, Series A*, **186**, 343-414.
- [4] Wang, Y., Hossain, A. M., and Zimmer, W. J. (2003), Monotone log-odds rate distributions in reliability analysis, *Comm. in Statis. Theo. and Meth.*, 32, 11, 2227-2244.
- [5] Wang, Y., Hossain, A. M., and Zimmer, W. J. (2008), Useful properties of the three-parameter Burr XII distribution, *Applied Statistics Research Progress, Editor: Mohammad Ahsanullah*, 11-20.
- [6] Zimmer, W. J., Wang, Y. and Pathak, P. K. (1998), Log-odds rate and monotone log-odds rate distributions, *Journal of Quality Technology*, **30**, 4, 376-385.