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Some Properties of Multivariate Skew-Normal Distribution, with Application to Strength-Stress model

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Abstract

In recent years, a large number of research works are appeared in the literature dealing with the properties and applications of the skew distributions. Skew distributions are shown to be flexible models for describing different kind of data. In the present study, we consider multivariate skew-normal distribution, and obtain some of its properties. These properties help us to explore the stress-strength model based on the multivariate skew-normal distribution.

Keywords: Linear combination, Multivariate skew-normal distribution, Skew-normal distribution, Stress-strength model.

1 Introduction

Let $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal density and cumulative distribution functions, respectively. Then, a random variable Z_λ is said to have a *standard skew-normal distribution* with parameter $\lambda \in \mathbb{R}$, denoted by $Z_\lambda \sim SN(\lambda)$, if its probability density function (pdf) is given by (Azzalini 1985, 1986)

$$\phi_{SN}(z;\lambda) = 2\phi(z)\Phi(\lambda z), \qquad z \in \mathbb{R}. \tag{1}$$

Azzalini and Dalla Valle (1996) presented the multivariate skew-normal distribution with the following pdf

$$\phi_{SN_n}(\mathbf{z};\boldsymbol{\Omega},\boldsymbol{\alpha}) = 2\phi_n(\mathbf{z};\boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}^T\mathbf{z}), \qquad \mathbf{z} \in \mathbb{R}^n, \tag{2}$$

where $\boldsymbol{\Omega}$ is $n \times n$ dimensional dispersion matrix, $\boldsymbol{\alpha} \in \mathbb{R}^n$ is vector of shape parameter and $\phi_n(\cdot;\boldsymbol{\Omega})$ denotes the pdf of the multivariate normal distribution with covariance matrix

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$\mathbf{\Omega}$. We denote by $\mathbf{Z} \sim SN_n(\mathbf{\Omega}, \boldsymbol{\alpha})$ and in special case that $\mathbf{\Omega} = \mathbf{I}_n$(Identity matrix), we denote by $\mathbf{Z} \sim SN_n(\boldsymbol{\alpha})$.

Azzalini and Dalla Valle (1996) presented representation of $\mathbf{Z} \sim SN_n(\mathbf{\Omega}, \boldsymbol{\alpha})$ as fallow. Let $\mathbf{Y} = (Y_1, \ldots, Y_n)^T$ and

$$\begin{pmatrix} Y_0 \\ \mathbf{Y} \end{pmatrix} \sim N_{n+1}\left(\mathbf{0}, \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{\Gamma} \end{pmatrix}\right), \tag{3}$$

where $\mathbf{\Gamma} = [\gamma_{i,j}]$ is $n \times n$ dimensional correlation matrix. Now if define $\mathbf{Z} = (Z_1, \ldots, Z_n)^T$ as

$$Z_i = \delta_i |Y_0| + \sqrt{1-\delta_i^2} Y_i, \tag{4}$$

where $\delta_i = \lambda_i / \sqrt{1+\lambda_i^2}$, $i = 1, \ldots, n$, then $Z_i \sim SN(\lambda_i)$ and $\mathbf{Z} \sim SN_n(\mathbf{\Omega}, \boldsymbol{\alpha})$, where $\mathbf{\Omega} = \mathbf{\Delta}\left(\mathbf{\Gamma} + \boldsymbol{\lambda}\boldsymbol{\lambda}^T\right)\mathbf{\Delta}$, $\boldsymbol{\alpha}^T = \frac{\boldsymbol{\lambda}^T \mathbf{\Gamma}^{-1} \mathbf{\Delta}^{-1}}{\sqrt{1+\boldsymbol{\lambda}^T \mathbf{\Gamma}^{-1} \boldsymbol{\lambda}}}$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^T$ and $\mathbf{\Delta} = diag\left\{\sqrt{1-\delta_1^2}, \ldots, \sqrt{1-\delta_n^2}\right\}$. In matrix form, we can represent

$$\mathbf{Z} = \boldsymbol{\delta} |Y_0| + \mathbf{\Delta} \mathbf{Y}. \tag{5}$$

In case $n = 2$, Gupta and Brown (2001) evaluated $P(Z_1 < Z_2)$ as follow

$$P(Z_1 < Z_2) = \frac{1}{\pi} \tan^{-1}\left(\frac{\delta_2 - \delta_1}{\sqrt{2 - \delta_1^2 - \delta_2^2}}\right) + \frac{1}{2}. \tag{6}$$

where $\delta_i = \lambda_i / \sqrt{1+\lambda_i^2}$, $i = 1, 2$. Let $X_i \stackrel{d}{=} \mu_i + \sigma_i Z_i$, $i = 1, 2$, where Z_1 and Z_2 represented as in (4). Mehrali and Asadi (2010) evaluated $P(X_1 < X_2)$ as follow

$$P(X_1 < X_2) = \Phi_{SN}\left(k/\sqrt{1+\delta^2}; \delta\right), \tag{7}$$

where $\Phi_{SN}(\cdot; \delta)$ is the cdf of $SN(\delta)$, $k = \frac{1}{\sigma}\frac{\mu_2 - \mu_1}{\sigma_1}$ and $\delta = \frac{a_1\delta_1 + a_2\delta_2}{\sigma}$, where δ_i, $i = 1, 2$ are as in (6), $\sigma^2 = a_1^2\left(1-\delta_1^2\right) + a_2^2\left(1-\delta_2^2\right)$, $a_1 = 1$ and $a_2 = -\frac{\sigma_2}{\sigma_1}$. Here we are interested in evaluation of the following model of which presented by Kotz et al. (2003) as

$$P(X_1 < X_2 < \cdots < X_n) \tag{8}$$

where $X_i \stackrel{d}{=} \mu_i + \sigma_i Z_i$, $i = 1, \ldots, n$, where $\mathbf{Z} = (Z_1, \ldots, Z_n)^T \sim SN_n(\mathbf{\Omega}, \boldsymbol{\alpha})$ with representation 5. For this purpose we study some properties of multivariate skew-normal distribution which help us to explore the stress-strength model based on the multivariate skew-normal distribution.

2 Some properties of multivariate skew-normal distribution

In this section, we present some properties of multivariate skew-normal distribution. These results help us to evaluate the stress-strength model based on the multivariate skew-normal distribution.

Let $Z_\lambda \sim SN(\lambda)$ independent of $\mathbf{W} \sim N_n(\mathbf{0}, \mathbf{\Sigma})$, where $N_n(\mathbf{0}, \mathbf{\Sigma})$ denotes the multivariate normal distribution.

a) If we define $\mathbf{Y}=\mathbf{H}^T\mathbf{W}+\mathbf{k}Z_\lambda$, then $\mathbf{Y}\sim SN_n(\boldsymbol{\Omega},\boldsymbol{\alpha})$, where $\mathbf{H}$ is $n\times n$ symmetric matrix, $\mathbf{k}\in\mathbb{R}^n$, $\boldsymbol{\Omega}=\mathbf{H}^T\Sigma\mathbf{H}+\mathbf{k}\mathbf{k}^T$, $\boldsymbol{\alpha}^T=\frac{\delta\mathbf{k}^T\boldsymbol{\Omega}^{-1}}{\sqrt{1-\delta^2\mathbf{k}^T\boldsymbol{\Omega}^{-1}\mathbf{k}}}$ and $\delta=\lambda/\sqrt{1+\lambda^2}$.

Let $Z_\lambda\sim SN(\lambda)$. Then

$$E\left[\Phi_n\left(\mathbf{k}Z_\lambda+\mathbf{u};\boldsymbol{\Sigma}\right)\right]=\Phi_{SN_n}\left(\mathbf{u};\boldsymbol{\Omega},\boldsymbol{\alpha}\right),$$

where $\Phi_n(\cdot;\boldsymbol{\Sigma})$ is the cdf of $N_n(\mathbf{0},\boldsymbol{\Sigma})$, $\Phi_{SN_n}(\cdot;\boldsymbol{\Omega},\boldsymbol{\alpha})$ is cdf of $SN_n(\boldsymbol{\Omega},\boldsymbol{\alpha})$, $\boldsymbol{\Omega}$ is same as lemma 2 part (c) and $\boldsymbol{\alpha}^T=-\frac{\delta\mathbf{k}^T\boldsymbol{\Omega}^{-1}}{\sqrt{1-\delta^2\mathbf{k}^T\boldsymbol{\Omega}^{-1}\mathbf{k}}}$.

Let define

$$\Psi_n(\mathbf{k},\mathbf{u},\boldsymbol{\Sigma})=\int_0^\infty\Phi_n(\mathbf{k}z+\mathbf{u};\boldsymbol{\Sigma})\,\phi(z)\,dz.$$

Then

$$\Psi_n(\mathbf{k},\mathbf{u},\boldsymbol{\Sigma})=\frac{1}{2}\Phi_{SN_n}\left(\mathbf{u};\boldsymbol{\Omega},\boldsymbol{\alpha}\right),$$

where $\boldsymbol{\Omega}$ is same as lemma 2 part (c) and $\boldsymbol{\alpha}^T=-\frac{\mathbf{k}^T\boldsymbol{\Omega}^{-1}}{\sqrt{1-\mathbf{k}^T\boldsymbol{\Omega}^{-1}\mathbf{k}}}$.

Let $\mathbf{Z}\sim SN_n(\boldsymbol{\Omega},\boldsymbol{\alpha})$ with representation 5 and $\mathbf{D}$ be an $(n-1)\times n$ matrix, then

$$P(\mathbf{DZ}<\mathbf{u})=\Phi_{SN_{n-1}}\left(\mathbf{u};\boldsymbol{\Omega}^*,\boldsymbol{\alpha}^*\right),$$

where where $\boldsymbol{\Omega}^*=\mathbf{D}^T\boldsymbol{\Omega}\mathbf{D}$ and $\boldsymbol{\alpha}^{*T}=\frac{\boldsymbol{\delta}^T\boldsymbol{\Omega}\mathbf{D}^{T^{-1}}}{\sqrt{1-\boldsymbol{\delta}^T\boldsymbol{\Omega}^{-1}\boldsymbol{\delta}}}$, where $\boldsymbol{\delta}$ is same as 4.

3 Stress-strength models in multivariate skew-normal distribution

Theorem 1. *Let $X_i\overset{d}{=}\mu_i+\sigma_iZ_i$, $i=1,\dots,n$, with representation as in (4). Then*

$$P(X_1<X_2<\cdots<X_n)=\Phi_{SN_{n-1}}\left(\mathbf{u};\boldsymbol{\Omega}^*,\boldsymbol{\alpha}^*\right),$$

where $\mathbf{u}=(u_1,\dots,u_{n-1})^T$, $u_i=\mu_{i+1}-\mu_i$, $\boldsymbol{\Omega}^*$ and $\boldsymbol{\alpha}^*$ are same as lemma 2 part (a) with

$$\mathbf{D}=\begin{bmatrix} a_1 & b_1 & & & \mathbf{0}\\ & a_2 & b_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & a_{n-1} & b_{n-1}\end{bmatrix}.$$

and $a_i=\sigma_i$ and $b_i=-\sigma_{i+1}$.

In special cases, we can find main results of Mehrali and Asadi (2010) and Gupta and Brown (2001) as in 6 and 7.

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