First Seminar on
Reliability Theory and its Applications
27-28, May 2015

Determining the Warranty Period Using Pitman Measure of Closeness

Mirfarah, E. [1] and Ahmadi, J. [2]

Department of Statistics, Ferdowsi University of Mashhad

Abstract

In this paper, we study the determination of the warranty period in view of a warranty policy where the manufacture accept to minimally repaired the failure product. To do this, the problem of predicting the time of minimal repair based on a progressive Type-II censored sample is considered. We utilize the property of Pitman measure of closeness and propose a method to find the closest predictor. Since, over-predication may be more important in a warranty problem, asymmetry loss is also considered in the probability of closeness.

Keywords: Pitman measure of closeness, Prediction, Warranty period, Progressively Type-II censored order statistics, Minimal repair.

1 Introduction

A warranty is a contractual agreement in which the manufacturer accept to rectify all failures occurring up to a given amount of time (warranty period) from the date of purchase. Manufacturers offer many types of warranties to promote their products such as repair, replacement or cash refund. Offering warranty leads to additional costs to the manufacturer, so choosing the best policy reduces the servicing costs of manufacturer. A detailed discussion of various issues related to warranties can be found in [5].

In this paper, we consider a policy where warranty is not renewed on product failure but it is minimally repaired. This means that, on repair, the failure rate of the item remains the same as just prior to failure. Such policies are suitable for complex and expensive products where repair typically involves a small part of the product. We are interested to predict the time of ith minimal repair to determine the perfect warranty

[1]elham_mirfarah@yahoo.com
[2]ahmadi-j@um.ac.ir

period. Moreover, the minimal repair times have the same joint distribution as record (R) values (see [2]), so to simplify the notation, in the rest of this paper, let R_i be the ith R, which has the same distribution as the ith minimal repair times.

Now, consider a life testing experiment involving n experimental units. Suppose m complete failure are to be observed, such that when the ℓth failure is observed, a_i items are randomly removed from the test. The vector $(a_1, \cdots, a_m)$ is fixed prior to the experiment. Let $X_{\ell:m:n}$ denotes the ℓth progressively Type-II censored (PTC) order statistic (OS) of the observed sample. We want to use this information to predict the minimal repair times. Statistical prediction play an important role in determining the warranty length. Many researches consider the prediction of a subset of ordered data based on an independent observed sample of ordered data and different methods are considered in the literatures, for more details see [3]. Here, the concept of Pitman's measure of closeness (PMC) is used to proposed a method for prediction.

The concept of PMC was introduced by [6] and faced a considerable attention in ordered data topics after [1]. For more review about the PMC, see the monograph by [4]. More formally, the PMC in prediction context is defined as follows.

Definition 1. *If T_1 and T_2 are two predictors of a random variable Y, then T_1 is a Pitman closer predictor than T_2, under loss function $L(\cdot,\cdot)$, if* $\Pr[L(T_1, Y) < L(T_2, Y)] \geq \frac{1}{2}$. *Moreover, let $\Lambda = \{T_1, T_2, ..., T_n\}$ be a non-empty class of predictors of Y. Then, T_i is the Pitman-closest predictor if, for every $T_j \in \Lambda$ such that $i \neq j$, we have* $\Pr[L(T_i, Y) < L(T_j, Y)] \geq \frac{1}{2}$.

Depending on the situation of problem, one can use different loss functions in the probability of PMC. Absolute loss function, i.e., $L(T, Y) = |T - Y|$, is the most common loss in PMC concept. However, in many warranty problem, under-prediction is more important than over-prediction or vice versa. So, apart from absolute loss function, in this paper, we consider the following loss function

$$L_1(T, Y) = \begin{cases} 0, & T < Y; \\ T - Y, & T > Y. \end{cases}$$

In the rest of this paper, we formulate the warranty issue as a prediction problem and study the PMC of OSs from current PTC sample to R values from a future sequence. Considering two loss functions in the probability of PMC, results have been compared.

2 Main result

Let $X_{\ell:m:n}$ denote the ℓth OS from a PTC sample with an absolutely continuous cumulative distribution function $F(\cdot)$ and probability density function $f(\cdot)$ and R_i be the ith R with the same parent distribution as $X_{\ell:m:n}$. Since PMC has the transitivity property in a class of ordered data, we consider the PMC of two adjacent OSs, i.e.,

$$\text{PMC}(X_{\ell:m:n}, X_{\ell+1:m:n}|R_i) = \Pr(|X_{\ell:m:n} - R_i| < |X_{\ell+1:m:n} - R_i|), \quad (1)$$

The exact expression for (1) is given as follows

$$\begin{aligned} \text{PMC} \quad & (X_{\ell:m:n}, X_{\ell+1:m:n}|R_i) = \Pr(X_{\ell:m:n} + X_{\ell+1:m:n} > 2R_i) \\ = \quad & \Pr(X_{\ell:m:n} > R_i) + \Pr(X_{\ell:m:n} < R_i, X_{\ell:m:n} + X_{\ell+1:m:n} > 2R_i) \\ = \quad & \sum_{t=1}^{\ell} c_{\ell-1}^{\mathcal{R}} a_t^{\mathcal{R}}(\ell) \left\{ \frac{1}{\gamma_t^{\mathcal{R}}} \left(\frac{1}{\gamma_t^{\mathcal{R}} + 1} \right)^{i+1} + \frac{1}{\gamma_{\ell+1}^{\mathcal{R}}} B(t, i) \right\}, \end{aligned}$$

where

$$B(t,i)=\int_0^1\int_y^1 u^{\gamma_t^{\mathcal{R}}-\gamma_{\ell+1}^{\mathcal{R}}-1}[\bar{F}(2F^{-1}(1-y)-F^{-1}(1-u))]^{\gamma_{\ell+1}^{\mathcal{R}}}\frac{\{-\log y\}^i}{i!}dudy.$$

PMC depends on the parent distribution of OSs. In the next section, we will find the result for exponential distribution.
Now, let us consider the problem of prediction using PMC with $L_1(\cdot,\cdot)$. Given a PTC sample, the PC probability to a R from a future independent sequence, under loss function $L_1(\cdot,\cdot)$, is given by

$$\Pr(L_1(X_{\ell:m:n},R_i)<L_1(X_{\ell+1:m:n},R_i))=\sum_{t=1}^{\ell+1}c_\ell^{\mathcal{R}}a_t^{\mathcal{R}}(\ell+1)\left\{\frac{1}{\gamma_t^{\mathcal{R}}}\left(\frac{1}{\gamma_t^{\mathcal{R}}+1}\right)^{i+1}\right\}.$$

It is important to note that, in the case of $L_1(\cdot,\cdot)$, PMC is non-parametric. In the next section, we will compare this results.

3 Example

We present our result in the previous section for the standard exponential in the case of absolute loss function. Then, the probability of closeness is compared with the results of non-parametric PMC.

Let the parent distribution be standard exponential, then $B(t,i)$ in the case of exponential is given as below

$$B(t,i)=\begin{cases}\frac{1}{\gamma_t^{\mathcal{R}}-2\gamma_{\ell+1}^{\mathcal{R}}}\left\{\left(\frac{1}{1+2\gamma_{\ell+1}^{\mathcal{R}}}\right)^{i+1}-\left(\frac{1}{1+\gamma_t^{\mathcal{R}}}\right)^{i+1}\right\}, & \gamma_t^{\mathcal{R}}\neq 2\gamma_{\ell+1}^{\mathcal{R}},\\[2ex] (i+1)\left(\frac{1}{1+2\gamma_{\ell+1}^{\mathcal{R}}}\right)^{i+2}, & \gamma_t^{\mathcal{R}}=2\gamma_{\ell+1}^{\mathcal{R}}.\end{cases}$$

Table 1 present the PMC of PTC OSs with censoring scheme $R=(20,0,0,0,0,0,0,0,0,0)$ to the first 6 Rs of future sequence. Table 2 is the non-parametric PMC when the loss function is $L_1(\cdot,\cdot)$.

Table 1: PMC for standard exponential. **Table 2**: Non-parametric PMC for $L_1(\cdot,\cdot)$.

	Table 1: i						Table 2: i					
ℓ	0	1	2	3	4	5	0	1	2	3	4	5
1	0.038	0.002	0.000	0.000	0.000	0.000	0.129	0.014	0.001	0.000	0.000	0.000
2	0.135	0.015	0.002	0.000	0.000	0.000	0.226	0.037	0.005	0.001	0.000	0.000
3	0.233	0.040	0.006	0.001	0.000	0.000	0.323	0.073	0.014	0.002	0.000	0.000
4	0.331	0.077	0.015	0.003	0.000	0.000	0.419	0.123	0.029	0.006	0.001	0.000
5	0.430	0.129	0.032	0.007	0.001	0.000	0.516	0.188	0.056	0.014	0.003	0.001
6	0.530	0.199	0.061	0.016	0.004	0.001	0.613	0.273	0.099	0.031	0.009	0.002
7	0.631	0.292	0.110	0.036	0.011	0.003	0.710	0.382	0.170	0.066	0.023	0.008
8	0.739	0.421	0.199	0.082	0.031	0.011	0.806	0.524	0.288	0.140	0.062	0.026
9	0.871	0.639	0.405	0.228	0.118	0.056	0.903	0.713	0.501	0.320	0.191	0.109

To find the Pitman closest OS for the specific R value, find the first ℓ which PMC is greater than 0.5. For example $X_{6:10:20}$ is the Pitman closest predictor for the first R when the loss function is absolute error. From Table 1 and 2, it can be seen that by ignoring the under-predict error, smaller OSs get closer to R value comparing with the time that we use absolute loss function.

References

[1] Balakrishnan, N., Davies, K. F., and Keating, J. P. (2009). Pitman closeness of order statistics to population quantiles. *Communications in Statistics-Simulation and Computation*, **38**, 802–820.

[2] Balakrishnan, N., Kamps, U., and Kateri, M. (2009). Minimal repair under a step-stress test. *Statistics and Probability Letters*, **79**, 1548–1558.

[3] Geisser, S. (1993). *Predictive Inference: An Introduction.* Chapman and Hall, New York.

[4] Keating, J. P., Mason, R. L. and Sen, P. K. (1993). *Pitman's Measure of Closeness: A Comparison of Statistical Estimators.* Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

[5] Murthy, D. N. P. and Blischke, W. R. (1992). Product warranty management-III: a review of mathematical models, *European Journal of Operational Research*, **63**, 1–34.

[6] Pitman, E. J. G. (1937). The "closest" estimate of statistical parameters. *Proceedings of the Cambridge Philosophical Society*, **33**, 212–222.