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Bayesian Inference for the Rayleigh Distribution Based on Record Ranked Set Samples

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Abstract

In this paper, we discuss the Bayesian estimation problem for the Rayleigh distribution based on upper record ranked set samples. The Bayes estimators are obtained with respect to two different loss functions. We also obtain the Bayes confidence intervals for the parameter of the Rayleigh distribution. Finally, we present a simulation study for the purpose of numerical comparison.

Keywords: General entropy loss function, Maximum likelihood estimator, Simulation.

1 Introduction

The record ranked set sampling scheme has been introduced recently by [2]. Here, we describe this sampling scheme, briefly, according to [2] as follows: Suppose that we have m independent sequences of continuous random variables. If $R_{i,i}$ denotes the i-th record value in the i-th sequence for $i = 1, ..., m$, then i-th sequence sampling is terminated when $R_{i,i}$ is observed. Then, the only available observations, which are called record ranked set sample (RRSS), include $R_{1,1}, \cdots, R_{m,m}$. These data can be minimal repair times of some reliability systems as mentioned in [2]. The Rayleigh distribution plays a key role in reliability analysis and therefore estimation of its parameter is important. In this paper, we consider the point and interval estimation problem for the Rayleigh distribution based on observed upper RRSSs. Main results are given in Section 2 and a simulation study is presented in Section 3.

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2 Main Results

We say that X has a Rayleigh distribution if its pdf is given by

$$f(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), \qquad x > 0, \quad \sigma > 0. \tag{1}$$

Let $\boldsymbol{U} = (U_{1,1}, \cdots, U_{m,m})$ be an upper RRSS from the Rayleigh distribution with pdf given in (1), then the likelihood function for the parameter σ given $\boldsymbol{u} = (u_{1,1}, \cdots, u_{m,m})$ is (see [2])

$$\begin{aligned} L(\sigma|\boldsymbol{u}) &= = \prod_{i=1}^{m} \frac{\{-\log(1 - F(u_{i,i}))\}^{i-1}}{(i-1)!} \, f(u_{i,i}) \\ &= \frac{\exp\left(-\frac{\sum_{i=1}^{m} u_{i,i}^2}{2\sigma^2}\right)}{\sigma^{m(m+1)} 2^{m(m-1)/2}} \prod_{i=1}^{m} \frac{u_{i,i}^{2i-1}}{(i-1)!}, \end{aligned} \tag{2}$$

where $u_{i,i} > 0$ for $i = 1, ..., m$ and $\sigma > 0$. The maximum likelihood estimator (MLE) of σ is readily obtained to be

$$\widehat{\sigma}_{ML} = \sqrt{\frac{\sum_{i=1}^{m} U_{i,i}^2}{m(m+1)}}. \tag{3}$$

For the Bayesian estimation, we take the conjugate prior

$$\pi(\sigma) \propto \sigma^{-2b-1} \exp(-a/2\sigma^2), \quad \sigma > 0, \tag{4}$$

where a and b are positive hyperparameters. Note that for $a = b = 0$, we arrive at the non-informative prior. Now, from Equations (2) and (4), the posterior distribution of σ, given $\boldsymbol{u}$, becomes

$$\pi(\sigma|\boldsymbol{u}) = \frac{2w^{b+m(m+1)/2} \exp(-w/\sigma^2)}{\Gamma\big(b + m(m+1)/2\big)\sigma^{2b+m(m+1)+1}}, \tag{5}$$

where $\Gamma(\cdot)$ is the complete gamma function, $W = \dfrac{a + \sum_{i=1}^{m} U_{i,i}^2}{2}$, and w is the observed value of W.

Let $\widehat{\sigma}$ be an estimator of σ, then the squared error loss (SEL) function is defined as $L_1(\sigma, \widehat{\sigma}) = (\widehat{\sigma} - \sigma)^2$. The Bayes estimator of σ under SEL function, based on RRSS, is the mean of the posterior density (5)

$$\widehat{\sigma}_{BS} = \frac{\Gamma\big(b + [m(m+1) - 1]/2\big)\sqrt{W}}{\Gamma\big(b + m(m+1)/2\big)}.$$

The SEL function is symmetric namely it assigns equivalent dimensions to underestimation and overestimation. But in many real situations, overestimation and underestimation have different consequences. Therefore, we consider a useful asymmetric loss function, called the general entropy loss (GEL) function, introduced by [1], which is defined as

$$L_2(\sigma, \widehat{\sigma}) = (\widehat{\sigma}/\sigma)^q - q \log(\widehat{\sigma}/\sigma) - 1, \quad q \neq 0.$$

The sign and magnitude of parameter q must be determined properly. The positive values of q cause the overestimation to get more serious than underestimation and vice versa. The Bayes estimator of σ under GEL function, based on RRSS, is

$$\widehat{\sigma}_{BG} = \left[E(\sigma^{-q}|\boldsymbol{U})\right]^{-1/q} = \left[\frac{\Gamma\big(b+[m(m+1)+q]/2\big)}{\Gamma\big(b+m(m+1)/2\big)}\right]^{-1/q}\sqrt{W}, \tag{6}$$

provided that $b+[m(m+1)+q]/2>0$.

Next, we want to find Bayesian confidence intervals for σ. From (5), we see that $Q_1 = w/\sigma^2|\boldsymbol{u} \sim G\big(b+m(m+1)/2, 1\big)$, where $G(\lambda_1,\lambda_2)$ stands for the gamma distribution with the shape parameter λ_1 and scale parameter λ_2. Let $\xi_\gamma(\lambda_1,\lambda_2)$ denote the upper γ-th quantile of $G(\lambda_1,\lambda_2)$, that is $P(V>\xi_\gamma(\lambda_1,\lambda_2))=\gamma$ where $V\sim G(\lambda_1,\lambda_2)$. Then, a $100(1-\alpha)\%$ two-sided equi-tailed Bayesian confidence interval (TEB CI) for σ is given by

$$\left(\sqrt{\frac{W}{\xi_{\alpha/2}(b+m(m+1)/2,1)}},\ \sqrt{\frac{W}{\xi_{1-\alpha/2}(b+m(m+1)/2,1)}}\right).$$

We also derive the highest posterior density intervals (HPDIs) for σ. For unimodal posterior distributions, the HPDIs are the same as their corresponding shortest credible intervals. Since the posterior pdf of σ given in (5) is unimodal, it can be easily verified that a $100(1-\alpha)\%$ HPDI for σ, given $W=w$, possesses the form (σ_L,σ_U) such that

$$\frac{\Gamma(A(b,m), w/\sigma_U^2, w/\sigma_L^2)}{\Gamma(b+m(m+1)/2)} = 1-\alpha, \quad \text{and} \quad \left(\frac{\sigma_U}{\sigma_L}\right)^{2A(b,m)+1} = e^{w(\sigma_L^{-2}-\sigma_U^{-2})},$$

where $A(b,m) = b+m(m+1)/2$ and $\Gamma(\nu,u_1,u_2) = \int_{u_1}^{u_2} t^{\nu-1}e^{-t}\mathrm{d}t$ is the generalized incomplete gamma function.

3 A simulation study

In this section, we performed a simulation in order to compare the point and interval estimators. In this simulation, we randomly generated $M=10000$ upper RRSSs of size $m=6$ from the Rayleigh distribution with $\sigma=1$. We considered 3 cases for the prior distribution described as follows:

Case I: Non-informative prior with $a=b=0$.

Case II: Informative prior with prior information $E(\sigma)=1$ =true value, and $Var(\sigma)=2$ and from (4), we have $a=0.9115$ and $b=1.1519$.

Case III: Informative prior with prior information $E(\sigma)=1$ and $Var(\sigma)=0.5$ which corresponds to $a=1.6989$ and $b=1.5663$.

We then obtained the MLEs and the Bayes estimators of σ under SEL and GEL (for $q=-2,\ 2$) functions, which are denoted by $\widehat{\sigma}_{ML}(i)$, $\widehat{\sigma}_{BS}(i)$ and $\widehat{\sigma}_{BG}(i)$, in the i-th iteration, respectively. The estimated risks (ERs) of the estimators were obtained using the relations $ER_S(\widehat{\sigma}_{BS}) = \frac{1}{M}\sum_{i=1}^{M}[\widehat{\sigma}_{BS}(i)-\sigma]^2$, and $ER_G(\widehat{\sigma}_{BG}) = \frac{1}{M}\sum_{i=1}^{M}[(\widehat{\sigma}_{BG}(i)/\sigma)^q - q\log(\widehat{\sigma}_{BG}(i)/\sigma)-1]$. We calculated the ER of each Bayes estimator according to its own loss function. For the MLEs, we calculated both kinds of ERs, i.e. ER_S and ER_G to compare them with their corresponding Bayes estimators. We also obtained 95% TEB CIs as well as 95% HPDIs for σ and calculated the coverage probabilities (CPs) and the average widths (AWs) of the CIs over 10000 replications. The results are presented in Table 1.

From Table 1, we observe that the ERs of the Bayes estimators, especially the ones obtained under the informative cases, are smaller than the ERs of the corresponding MLEs which reveals the superiority of the Bayesian methods as compared with the likelihood ones. Moreover, Case III contains the smallest ERs and the shortest CIs which is quite reasonable as Case III has the smallest prior variance and therefore is the most informative case. We also observe that the HPDIs are shorter than their corresponding TEB CIs.

Table 1: The results of the simulation.

	ER_S	ER_G		TEB CI		HPDI	
		$q=-2$	$q=2$	AW	CP	AW	CP
MLE	0.0119	0.0260	0.0240	–	–	–	–
Case I	0.0124	0.0248	0.0240	0.4437	0.9500	0.4364	0.9493
Case II	0.0114	0.0242	0.0235	0.4243	0.9484	0.4177	0.9426
Case III	0.0110	0.0232	0.0227	0.4201	0.9510	0.4137	0.9446

References

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