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Estimation for the Weighted Exponential Distribution Using the Probability Weighted Moments Method

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Abstract

In this paper, we focus on the problem of estimation for the scale and shape parameters of the weighted exponential distribution. The probability weighted moments method has been developed for estimating the parameters. A real data example ends the paper.

Keywords: Maximum likelihood estimator, Probability weighted moment method, Weighted exponential distribution.

1 Introduction

The weighted exponential (WE) distribution was first introduced by [3] and has the following probability density function (pdf)

$$f(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1-e^{-\lambda\alpha x}), \quad x>0, \quad \alpha>0, \quad \lambda>0. \tag{1}$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x) = 1 - \frac{\alpha+1}{\alpha}e^{-\lambda x} + \frac{1}{\alpha}e^{-\lambda(1+\alpha)x}, \qquad x>0.$$

The WE distribution can be applicable in reliability and therefore estimation of its parameters are important in this field. Recently, Dey et al. [1] investigated different methods of estimation for this distribution, including the moment, maximum likelihood (ML), weighted least-squares and percentile methods. But they did not consider one of the well-known methods known as the probability weighted moments (PWM) method. In what follows, we review the procedure of finding the moment and ML estimators of the parameters and then discuss how to obtain PWM estimators. Section 2 contains the main results and a real data example is given in Section 3.

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2 Main Results

Let $X_1, \cdots, X_n$ be a random sample of size n from the WE distribution with pdf given in (1). In this section, we discuss the moment, ML and PWM methods for estimation of the parameters.

2.1 Moment Estimation

In this method, the estimators of the parameters are obtained by equating the population moments with the sample moments. For the WE distribution, the moment estimators (MEs) of the parameters are (see [3])

$$\widehat{\alpha}_{ME} = \frac{-(\overline{X}^2 - 2S^2) + \sqrt{(\overline{X}^2 - 2S^2)^2 - 2(\overline{X}^2 - S^2)(\overline{X}^2 - 2S^2)}}{\overline{X}^2 - S^2},$$

and $\widehat{\lambda}_{ME} = \frac{1}{\overline{X}}\left(1 + \frac{1}{1+\widehat{\alpha}_{ME}}\right)$, where $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$ is the sample mean and $S^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \overline{X}^2$ is the sample variance. It can be proved that the MEs exist and are feasible if and only if $S^2 < \overline{X}^2 < 2S^2$, see [3].

2.2 Maximum Likelihood Estimation

The log likelihood function for the WE distribution, given an observed random sample of size n, is given by
$\ell(\alpha;\lambda,x) = n\ln(\alpha+1) - n\ln\alpha + n\ln\lambda - \lambda\sum_{i=1}^n x_i + \sum_{i=1}^n \ln(1 - e^{-\lambda\alpha x_i})$.

The ML estimators will be obtained by maximization of the log likelihood function with respect to the parameters. Upon differentiating the log likelihood function with respect to the parameters and equating them with zero, we have

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha+1} - \frac{n}{\alpha} + \lambda\sum_{i=1}^n \frac{x_i e^{-\lambda\alpha x_i}}{(1 - e^{-\lambda\alpha x_i})} = 0,\\ \frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^n x_i + \alpha\sum_{i=1}^n \frac{x_i e^{-\lambda\alpha x_i}}{1 - e^{-\lambda\alpha x_i}} = 0.\end{aligned}$$

Numerical techniques may be applied to solve the equations.

2.3 Probability weighted moments method

The PWM method was first introduced by [2]. For an arbitrary random variable X with cdf $F(x)$, the probability weighted moment of order (l,k,r) is defined as

$$M_{l,k,r} = E[X^k\{F(X)\}^k\{1 - F(X)\}^r],$$

where l, k and r are real numbers. Clearly, the quantities $M_{l,0,0}, l = 1, 2, ...$ are the usual noncentral moments. In the context of estimation, it is preferable to use either $M_{1,k,0}, k = 0, 1, 2, ...$ or $M_{1,0,r}, r = 0, 1, 2, ...$ depending on the structure of the cdf of X, see [4] for more related details. Landwehr et al. [5] emphasized that an unbiased estimator of $M_r \equiv M_{1,0,r}$, when r is a nonnegative integer number, based on a random sample of size n, is

$$\widehat{M_r} = \frac{1}{n}\sum_{i=1}^{n} X_i \binom{n-i}{r} \Big/ \binom{n-1}{r}.$$

Therefore, one can obtain the PWM estimators of the unknown parameters by equating M_r with $\widehat{M_r}$ for $r = 0, 1, ..., s$ where $s+1$ is the number of parameters. For the WE distribution, $M_0 = E(X) = \lambda^{-1}(1 + \frac{1}{1+\alpha})$ and

$$\begin{aligned} M_1 &= E[X\{1-F(X)\}] = \int_0^\infty x\left(\frac{\alpha+1}{\alpha}e^{-\lambda x} - \frac{1}{\alpha}e^{-\lambda(\alpha+1)x}\right) \\ &\quad \times \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1-e^{-\lambda\alpha x})\mathrm{d}x \\ &= \frac{\alpha+1}{\lambda\alpha^2}\left(\frac{\alpha+1}{4} - \frac{1}{\alpha+2} + \frac{1}{4(\alpha+1)^2}\right). \end{aligned}$$

Thus, we can obtain the PWM estimators of λ and α by solving the equations: $M_0 = \overline{X}$ and $M_1 = \frac{1}{n(n-1)}\sum_{i=1}^{n}(n-i)X_i$, simultaneously. From these equations, we can see that, for a given α, the PWM estimator of λ is

$$\widehat{\lambda}_{PWM}(\alpha) = \frac{1}{\overline{X}}\left(1 + \frac{1}{1+\alpha}\right), \tag{2}$$

and the PWM estimator of α can be obtained as a solution of the following fixed-point type equation

$$\frac{(\alpha+1)^2}{(\alpha+2)\alpha^2}\left(\frac{\alpha+1}{4} - \frac{1}{\alpha+2} + \frac{1}{4(\alpha+1)^2}\right) = \frac{1}{n(n-1)\overline{X}}\sum_{i=1}^{n}(n-i)X_i.$$

Once we get the PWM estimator of α, the PWM estimator of λ can be obtained from (2).

3 A real data example

Here, we consider a real data set, reported by [3, page 632], which is the marks of the slow pace students in Mathematics in the final examination in 2003. For these data, the Moment, ML and the PWM estimators of the parameters are: $\widehat{\alpha}_{ME} = 0.4384$, $\widehat{\lambda}_{ME} = 0.0655$, $\widehat{\alpha}_{MLE} = 0.2919$, $\widehat{\lambda}_{MLE} = 0.0685$, $\widehat{\alpha}_{PWM} = 0.0007764$, $\widehat{\lambda}_{PWM} = 0.0772$.

The following codes in R 3.1.2 were used to find the PWM estimators of the parameters.

```
library(nleqslv)
x=c(29,25,50,15,13,27,15,18,7,7,8,19,12,18,5,21,15,86,21,15,14,
39,15,14,70,44,6,23,58,19,50,23,11,6,34,18,28,34,12,37,4,60,20,
23,40,65,19,31)
z=c()
for(i in 1:n)  z[i]=(n-i)*x[i]
MODEL=function(u){
v=numeric(1)
v[1]=(u[1]+1)^2/(u[1])^2/(u[1]+2)*( (u[1]+1)/4-1/(u[1]+2)
+1/4/(u[1]+1)^2)-1/sum(x)/(n-1)*sum(z)
v
```

```
}
RES=nleqslv(c(2),MODEL)
alph=RES$x[1]
lamb=1/mean(x)*(1+1/(alph+1))
```

References

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