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Stress-strength system with non-identical exponentiated exponential distribution

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Abstract

A multicomponent stress-strength system is considered, while the stress and the strength system have non-identical exponentiated exponential distributions with different parameters. The estimation of stress-strength reliability parameter is studied.

Keywords: Stress-strength reliability, Uniformly minimum variance unbiased estimator, Maximum likelihood estimator

1 Introduction

The problem of increasing reliability of any system is now a well-recognized and rapidly developing branch of engineering. Stress-strength reliability the probability that the random variable X (stress) is exceeded by its strength which is a realization of a random variable Y which is equal to $R := P(X < Y)$. The problem of estimation of R has been discussed in the literature extensively. Multicomponent stress-strength reliability also has been studied by several authors, see for examples, Bhattacharyya and Johnson (1974), Pandey et al. (1992) and Eryilmaz (2008b). Nguimkeu et al. (2014) proposed a procedure to obtain accurate confidence intervals for the stress-strength reliability $R = P(X > Y)$ when (X, Y) is a bivariate normal distribution with unknown means and covariance matrix. Cha and Finkelstein (2015) studied a dynamic stress-strength model under external shocks.

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2 Main aim of the paper

In the present paper, we consider the following multicomponent stress-strength:

- A parallel system of n_1 components having stress following n_1 independently and non-identically distributed random variables X_i for $i = 1, \ldots, n_1$ with cdf $F_i(x) = [1 - \exp(-\gamma x)]^{\alpha(i)}$, where γ and $\alpha(i)$ are positive constants.
- The strengths of the components are independent but non-identical random variables Y_j for $j = 1, ..., n_2$ with cdf $G_j(y) = [1 - \exp(-\beta y)]^{\nu(j)}$, where β and $\nu(j)$ are positive constants.

The aforementioned cdfs F_i and G_j are known as exponentiated exponential distribution in the literature, see for example Gupta and Kundu (2001). As one can see, we can assume that the stress system consist n_1 parallel systems where the ith system contains $\alpha(i)$ parallel components with independent and identical cdf $F(x) = 1 - \exp(-\gamma x)$, for $i = 1, \ldots, n_1$. Under these assumptions, we find

$$R = P\left(\max_{1\le j\le n_2} Y_j > \max_{1\le i\le n_1} X_i\right) = P(W > Z) = \int_0^\infty F_Z(w) g_W(w)\, dw, \tag{1}$$

where $W = \max_{1\le j\le n_2} Y_j$ and $Z = \max_{1\le i\le n_1} X_i$. Then, we have

$$f_Z(z) = \gamma e^{-\gamma z} \sum_{i=1}^{n_1} \alpha(i)(1 - e^{-\gamma z})^{\sum_{i=1}^{n_1} \alpha(i) - 1} \tag{2}$$

and

$$g_W(w) = \beta e^{-\beta w} \sum_{j=1}^{n_2} \nu(j)(1 - e^{-\beta w})^{\sum_{j=1}^{n_2} \nu(j) - 1}. \tag{3}$$

By substituting (2) and (3) into (1), and doing some calculations, we obtain

$$R = \int_0^1 \left[1 - \left(1 - u^{\frac{1}{\sum_{j=1}^{n_2} \nu(j)}}\right)^{\frac{\gamma}{\beta}}\right]^{\sum_{i=1}^{n_1} \alpha(i)} du. \tag{4}$$

In what follows, we will study the estimation of R in (4).

3 Estimation of R

We obtain two common point estimators of R, namely MLE and UMVUE.

MLE: Let $Z_1, \ldots, Z_n$ be a random sample of size n from Z with pdf in (2) and $W_1, \ldots, W_m$ be a random sample of size m from each distributed as W with pdf in (3). Using the invariance property of MLE, the MLE of R is given by

$$\widehat{R}_M = \int_0^1 \left[1 - \left(1 - x^{\frac{U}{m}}\right)^{\frac{\gamma}{\beta}}\right]^{\frac{n}{T}} dx, \tag{5}$$

where $U = -\sum_{s=1}^{m} \log(1 - e^{-\beta W_s})$ and $T = -\sum_{r=1}^{n} \log(1 - e^{-\gamma Z_r})$.

UMVUE: To obtain the UMVUE of R noting that U and T are complete sufficient statistics for $\sum_{j=1}^{n_2} \nu(j)$ and $\sum_{i=1}^{n_1} \alpha(i)$ and are independent and each distributed as $\Gamma\left(m, \sum_{j=1}^{n_2} \nu(j)\right)$ and $\Gamma\left(n, \sum_{i=1}^{n_1} \alpha(i)\right)$, respectively. By doing some calculations, we have

$$\begin{aligned}\widehat{R}_U &= \int_{-\frac{1}{\gamma}\ln(1-e^{-t})}^{\infty} \frac{(m-1)\beta e^{-\beta w_1}}{u(1-e^{-\beta w_1})} \left(1 + \frac{\ln(1-e^{-\beta w_1})}{u}\right)^{m-2} \left(1 + \frac{\ln(1-e^{-\gamma w_1})}{t}\right)^{n-1} dw_1 \\ &= (m-1)\sum_{r=0}^{n-1} \binom{n-1}{r} \frac{1}{t^r} \int_{1+\frac{1}{u}\ln\left(1-(1-e^{-t})^{\frac{\gamma}{\beta}}\right)}^{1} s^{m-2} \left(\ln(1-(1-e^{u(s-1)})^{\frac{\gamma}{\beta}})\right)^r ds. \qquad (6)\end{aligned}$$

if $-\frac{1}{\gamma}\ln(1-e^{-t}) \geq -\frac{1}{\beta}\ln(1-e^{-u})$ and

$$\begin{aligned}\widehat{R}_U &= \int_{-\frac{1}{\beta}\ln(1-e^{-u})}^{\infty} \frac{(m-1)\beta e^{-\beta w_1}}{u(1-e^{-\beta w_1})} \left(1 + \frac{\ln(1-e^{-\beta w_1})}{u}\right)^{m-2} \left(1 + \frac{\ln(1-e^{-\gamma w_1})}{t}\right)^{n-1} dw_1 \\ &= (m-1)\sum_{r=0}^{n-1} \binom{n-1}{r} \frac{1}{t^r} \int_{0}^{1} s^{m-2} \left(\ln(1-(1-e^{u(s-1)})^{\frac{\gamma}{\beta}})\right)^r ds, \qquad (7)\end{aligned}$$

if $-\frac{1}{\gamma}\ln(1-e^{-t}) < -\frac{1}{\beta}\ln(1-e^{-u})$.

The performance of MLE and UMVUE are compared in the next section, by using the mean squared error (MSE), through generated many sample sizes by using simulation technique.

4 Numerical studies and conclusions

Our goal in this section is to compare the presented estimators, numerically when R changes from 0.01 to 0.99. For this purpose, we assume that $\gamma = 2$, $n_1 = n_2 = 5$, $\alpha(i) = (1+\theta_1)^{i-1}$ for $i = 1, \ldots, n_1$, $\nu(j) = (1+\theta_2)^{j-1}$ for $j = 1, \ldots, n_2$ and values (5,5), (5,10), (10,5),(10,10) for (m, n). we consider three different values for (θ_1, θ_2) as (1,1) and (1, 3). For each of them, we obtain MSE and bias of UMVU and ML estimators. The results are displayed in Figures 1 and 2.

References

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