First Seminar on
Reliability Theory and its Applications
27-28, May 2015

The Generalized Joint Signature for Systems with Shared Components

Zarezadeh, S. [1] and Mohammadi, L. [2]

Department of Statistics, Shiraz University

Abstract

The concept of joint signatures which first defined by Navarro et al. [1] are useful tools for investigating the reliability of two systems with shared components. When several coherent systems share some components and the components have independent and identically distributed (i.i.d.) lifetimes, we obtain a pseudo-mixture representation for the joint distribution of the lifetimes of the systems based on a general notion of joint signatures. We present an R program to find the mentioned joint signature for any number of systems and components.

Keywords: Coherent system, Order statistic, Signature.

1 Introduction

This paper is a continuation of Navarro et al. [1] in which the joint behavior of several systems with at least one shared component is investigated. The joint distribution function of the system lifetimes is obtained generally in a theorem and then various illustrated examples are also provided.

An example of systems with shared components, which is used often, is in networked computing in which a server is used at the same time with several computers. A central server stores almost all of the files for the department's computers. If the central server breaks, some of the computers will not work at all and some will have limited capabilities. The performance of any given pair of PCs will depend on the performance of the shared components and that of its own individual components. Navarro et al. [1] found the joint distribution of the lifetimes of the computers when all component lifetimes are i.i.d.

[1] s.zarezadeh@shirazu.ac.ir
[2] mohammadileila@yahoo.com

random variables. They used a measure called signature which was defined previously in Samaniego [2], [3]. The signature of a coherent system with n i.i.d. components is defined as the probability vector $s = (s_1, s_2, ..., s_n)$, where s_i is the probability that the system fails when the i-th component fails. Hence, $s_i = P(T = X_{i:n})$, for $i \in \{1, 2, ..., n\}$, where T is the system's lifetime and $X_{1:n}, X_{2:n}, ..., X_{n:n}$ are the order statistics corresponding to the n component lifetimes. A system's signature is useful and interesting specially because it is distribution-free. Using the signatues is an efficient method to find the precise features that influence the performance of a system's design.

Navarro et al. [1] proposed joint signatures by considering two systems sharing some components. We want to generalize this concept. Suppose we have a cluster of systems, some sharing components with some others. Then, we study how to define a very general notion of joint signatures. The following scenario can be imagined. Suppose we have three systems. Then, we can split the case into components that are shared by all three, components that are common only for 1 and 2, components that are common only for 1 and 3, and components that are common only for 2 and 3. By this setting, we obtain a pseudo-mixture representation for the joint distribution function of the lifetimes of three systems based on distributions of order statistics of component lifetimes and then develop a general notion of joint signatures. This will then generalize to more than three systems in the cluster.

In this article, we obtain the joint distribution of lifetimes of more than two cohorent systems with n i.i.d. components. The definition of signature in our representations is related to the one defined in Navarro et al. [1], but it is more general. We present R programs to obtain the multidimensional distribution of the lifetimes of at least three systems. Our programs give a signature matrix which provides, in fact, the coefficients of the distribution functions of the order statistics of the n iid component lifetimes in the representation of the multidimensional distribution function of the systems.

2 Main results

Suppose that $X_1, X_2, \dots, X_n$ are non-negative independent random variables with common distribution function F. Consider three systems with lifetimes $T_1 = \phi_1(Y_1, Y_2, \dots, Y_{n_1})$, $T_2 = \phi_2(Z_1, Z_2, \dots, Z_{n_2})$, and $T_3 = \phi_3(W_1, W_2, \dots, W_{n_3})$. Let $\{Y_1, Y_2, \dots, Y_{n_1}\}$, $\{Z_1, Z_2, \dots, Z_{n_2}\}$, and $\{W_1, W_2, \dots, W_{n_3}\}$ be subsets of $\{X_1, X_2, \dots, X_n\}$. If we denote the joint distribution function of $\mathbf{T} = (T_1, T_2, T_3)$ by $G(t_1, t_2, t_3) = \mathbb{P}(T_1 \le t_1, T_2 \le t_2, T_3 \le t_3)$, the following theorem is obtained.

Theorem 1. *Let $\{i_1, i_2, i_3\}$ be a permutation of $\{1, 2, 3\}$. Then the joint distribution function of $\mathbf{T}$, denoted by G, is written as*

$$G(t_{i_1}, t_{i_2}, t_{i_3}) = \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{i=1}^{n} s_{i,j,k}^{(i_1, i_2, i_3)} F_{i:n}(t_{i_1}) F_{j:n}(t_{i_2}) F_{k:n}(t_{i_3})$$

$$\text{for} \qquad t_{i_1} \le t_{i_2} \le t_{i_3}. \tag{1}$$

Proof. The proof is removed because of the restriction on the number of the pages of the paper. □

If we have m coherent systems with respective lifetimes $T_1, T_2, \dots, T_m$, then, for $t_{i_1} \le$

$t_{i_2} \leq \cdots \leq t_{i_m}$,

$$G(t_{i_1}, t_{i_2}, \ldots, t_{i_m}) = \sum_{i_1=1}^{n} \sum_{i_2=0}^{n-i_1} \cdots \sum_{i_m=0}^{n-\sum_{j=1}^{m-1} i_j} c_{i_1,i_2,\ldots,i_m} \prod_{j=1}^{m} F^{i_j}(t_{i_j}),$$

where $c_{i_1,i_2,\ldots,i_m}$ are integers which do not depend on the underlying distribution function F.

In the following remark, we present a matrix form for the joint distribution function of $\mathbf{T}$. We use the notation $B = \{b_i\}_n$ for a matrix B with n rows B_i, $i = 1, ..., n$. Define $\mathbf{a}'_{t_{i_1}} = (F_{1:n}(t_{i_1}), \ldots, F_{n:n}(t_{i_1}))$, and

$$\mathbf{a}'_{t_{i_j}} = (F_{0:n}(t_{i_j}), F_{1:n}(t_{i_j}), \ldots, F_{n:n}(t_{i_j})),\ j = 2, 3.$$

The joint distribution function of $\mathbf{T}$ in (1) can be rewritten as $G(t_1, t_2, t_3) = \mathbf{a}'_{t_{i_1}} W$, with $W = \{\mathbf{a}'_{t_{i_2}} S_l^{(i_1,i_2,i_3)} \mathbf{a}_{t_{i_3}}\}_n$ and $S_l^{(i_1,i_2,i_3)} = \{s_{l,j,k}^{(i_1,i_2,i_3)}\}_{(n+1)\times(n+1)}$, $l = 1, 2, \ldots, n$. Define $A = \{\overbrace{0, \ldots, 0}^{in \text{ times}}\ \mathbf{a}'_{t_2}\ \overbrace{0, \ldots, 0}^{n(n-i) \text{ times}}\}_{n+1}$ and $S^{(i_1,i_2,i_3)} = \{S_i^{(i_1,i_2,i_3)}\}_n$, then, $G(t_1, t_2, t_3) = \mathbf{a}'_{t_1} A S \mathbf{a}_{t_3}$.

Example 1. Let X_1, X_2, X_3 be i.i.d. random variables with common distribution function F. Consider three coherent systems with lifetimes $T_1 = \max\{X_1, X_2\}$, $T_2 = \max\{X_1, X_3\}$, and $T_3 = \max\{X_1, X_4\}$. Thus the systems have one shared component. Then the joint distribution function of T_1, T_2, and T_3, for $t_1 \leq t_2 \leq t_3$, is written as

$$G(t_1, t_2, t_3) = F_{2:2}(t_1) F_{1:1}(t_2) F_{1:1}(t_3).$$

The signature of order 4 of $X_{2:2}$ and $X_{1:1}$ are obtained, respectively, as $(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ and $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Hence, by some calculations, the joint signature $S_1^{(1,2,3)}$ is obtained as a 4×5 zero matrix and $S_2^{(1,2,3)}$, $S_3^{(1,2,3)}$ and $S_4^{(1,2,3)}$ are 4×5 matrices with the first row and first column all zero's and the rest a constant which are respectively $\frac{1}{96}$, $\frac{1}{48}$, and $\frac{1}{32}$.

Then, by letting

$$\mathbf{a}'_{t_1} = (F_{1:4}(t_1), F_{2:4}(t_1), F_{3:4}(t_1), F_{4:4}(t_1)),$$

and

$$\mathbf{a}'_{t_i} = (F_{0:4}(t_i), F_{1:4}(t_i), F_{2:4}(t_i), F_{3:4}(t_i), F_{4:4}(t_i)), \quad i = 2, 3,$$

one obtains $G(t_1, t_2, t_3) = \mathbf{a}'_{t_1} W$, where $W = \{\mathbf{a}'_{t_2} S_n^{(1,2,3)} \mathbf{a}_{t_3}\}_n$.

For any permutation (i_1, i_2, i_3) of $\{1, 2, 3\}$ such that $t_{i_1} \leq t_{i_2} \leq t_{i_3}$, the distribution function G is followed same as the case where $t_1 \leq t_2 \leq t_3$.

Example 2. *Consider three systems with lifetimes* $T_1 = \max\{X_1, X_2\}$, $T_2 = \max\{X_2, X_3\}$, *and* $T_3 = \max\{X_3, X_4\}$.

Using the same procedure as the previous example, we obtain $S_1^{(1,3,2)}$ *which is a* 5×5 *zero matrix. Also,* $S_2^{(1,3,2)}, S_3^{(1,3,2)}$ *and* $S_4^{(1,3,2)}$ *are* 5×5 *matrices with all elements zero except for the last three rows of the first column which are respectively* $(\frac{1}{36}, \frac{1}{18}, \frac{1}{12})$, $(\frac{1}{18}, \frac{1}{9}, \frac{1}{6})$, *and* $(\frac{1}{12}, \frac{1}{6}, \frac{1}{4})$.

We have written an R pacakge to find the joint distribution of $T_1, ..., T_m$, for general $m > 2$. Our package takes as input m matrices, each $n \times n$, containing the relation between each T_i and $(X_1, ..., X_n)$, $i = 1, ..., m$. Then by running our program, we receive the following outputs for each permutation of $1, ..., m$: The joint distribution function of the systems in terms of $F(t_i)$'s, $i = 1, ..., m$, and also in terms of $F_{j:n}(t_i)$'s, $i = 1, ..., m,$, $j = 1, ..., n$, a matrix of the coefficients of $F_{j:n}(t_i)$'s, $i = 1, ..., m,$, $j = 1, ..., n$, and the general joint signature matrix S, which is an $(m+1)(n+1) \times (n+1)$ matrix. Hence, for example 1, this matrix binds all the matrices $S_i^{(1,2,3)}$'s, $i = 1, ..., 4$ in rows and it is a 20×5 matrix.

References

[1] Navarro, J., Samaniego, F. J. and Balakrishnan, N. (2010), The joint signature of coherent systems with shared components, *Journal of Applied Probability*, **47**(1), 235-253.

[2] Samaniego, F. J. (1985), On closure of the IFR class under formation of coherent systems, *IEEE Transactions on Reliability*, **34**, 69-72.

[3] Samaniego, F. J. (2007), *System Signatures and Their Applications in Reliability Engineering*, New York, Berlin: Springer, 2007.